Derived categories and the geometry of projective varieties

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Table of Contents

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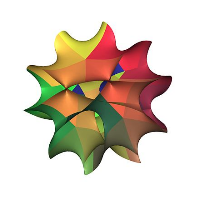
Let X be the zero-locus in $\mathbb{P}^4$ of

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0.$$

It is a **Calabi–Yau 3-fold** ($K_X \equiv 0$) which is called **Fermat quintic 3-fold**.



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The definition

- *Objetcs*: bounded complexes of coherent sheaves

$$\cdots \to 0 \to E^i \to \ldots \to E^{i+n} \to 0 \to \ldots$$

- *Morphisms*: finite sequences of roofs

$$\begin{array}{ccccccccc} & & D_1 & & \ldots & & D_n & & \\ & \overset{\text{quis}}{\swarrow} & & \searrow & & \overset{\text{quis}}{\swarrow} & & \searrow & \\ A & & & & A_1 \quad \ldots \quad A_{n-1} & & & & B. \end{array}$$

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It is **triangulated**:

- We can shift objects ($E[1]$);
- *Exact triangles*

$$A \to B \to C \to A[1]$$

play the same role as short exact sequences in $\mathrm{Coh}(X)$.

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Good news

There are cases where $\mathrm{D}^b(X)$ proves to be a strong invariant:

Theorem (Bondal and Orlov, 2001)

Let X be a smooth projective variety such that K_X is either ample or antiample. Let Y be a smooth projective variety such that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. Then $X \cong Y$.

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Example (in the negative)

Let X be the Fermat quintic 3-fold ($K_X \equiv 0$).

The theorem **does not** apply!

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We need to add more structure to $\mathrm{D}^b(X)$!

Table of Contents

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$$H^0(\mathbf{Inj}(X)) = \mathrm{D}^b(X).$$

(B) **Stability conditions:** endow $\mathrm{D}^b(X)$ with a good notion of *stability* which allows to cut out of $\mathrm{D}^b(X)$ moduli spaces with geometric meaning.

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- Give a rigorous definition.
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In the rest of the presentation we focus on (A) and (B)!

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Mirror Symmetry: a study case

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Idea:

X and $\check{X}$ are *compactifications* of different string theories (type A and B, resp.).

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Rough idea:

$\mathrm{DFuk}^\pi(\check{X})$ is the **Fukaya derived category**: homotopy category of an A_∞ category $\mathrm{D}_\infty\mathrm{Fuk}^\pi(\check{X})$ whose objects are Lagrangian submanifolds and morphisms are intersection numbers.

Whereof one cannot speak, thereof one must be silent.

L. Wittgenstein, Tractatus logico-philosophicus

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- If X is a CY 3-fold, then $\mathrm{D}^b(X)$ has (conjecturally!) at least two enhancements:
  - the dg category $\mathbf{Inj}(X)$;
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In relation to the first item the following is natural:

Conjecture (Bondal–Larsen–Lunts)

If X is a smooth projective variety, then $\mathrm{D}^b(X)$ has a unique enhancement.

Table of Contents

Def. (dg categories)

A **differential graded (dg) category** is a k-linear category (k a comm. ring) such that

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Example: injective resolutions

We have already see that if X is a smooth projective scheme, then $\mathbf{Inj}(X)$ is a dg category.

It is actually **pretriangulated**! ...roughly:

$$H^0(\text{dg-cat}) \cong \text{triang. cat.}$$

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A **dg functor** $\mathsf{F}\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor such that

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We then have the following constructions:

- Given a dg functor $\mathsf{F}\colon \mathcal{C}_1 \to \mathcal{C}_2$, we can compute

$$H^0(\mathsf{F})\colon H^0(\mathcal{C}_1) \to H^0(\mathcal{C}_2).$$

- A dg functor F is a **quasi-equivalence** if
  - Φ_{F} is a quasi-isomorphism;
  - $H^0(\mathsf{F})$ is an equivalence.

Drinfeld, Kontsevich, Keller,...: one can form the following

$$\mathrm{Hqe} := \text{dg-Cat}[\text{q-eq}^{-1}]$$
$$= \text{loc. wrt quasi-equiv.}$$

In practice:

- *Objetcs*: dg categories;
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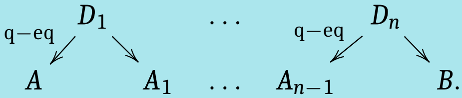

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An **enhancement** of a triangulated category $\mathcal{T}$ is a part $(\mathcal{C}, \mathsf{F})$ where $\mathcal{C}$ is a pretriang. dg cat. and $\mathsf{F}\colon H^0(\mathcal{C}) \to \mathcal{T}$ is an equivalence.

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Def. (uniqueness of enhancements)

A triang. cat has a **unique enhancement** if any two such are isomorphic in Hqe.

BLL Conjecture: proven by Lunts–Orlov (JAMS, 2010). Additional improvements by: Canonaco–S., Antieau, Genovese. The following covers additional conj./open problems:

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Canonaco–Ornaghi–S.

By old and recent results of the three of us, the thm above applies to A_∞ categories as well, covering the case of $\mathrm{D}_\infty\mathrm{Fuk}^\pi(\check{X})$.

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$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$$

such that E_i/E_{i-1} is semistable with respect to

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(B) For any $0 \neq E \in \mathcal{A}$ there is a *Harder–Narasimhan filtration* with factors which are semistable with respect to

$$\mu := -\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}.$$

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Warning:

$\mathrm{Stab}_\Lambda(X) \neq \emptyset$ stricking and difficult problem! Expecially when $K_X \equiv 0$ and the dim grows.

Case by case

Theorem (Beauville, Bogomolov)

Assume X smooth proj. with $c_1 = 0$. Up to a finite étale map, X is isomorphic to a product varieties of the following types:

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- Abelian variety;

Definition

$X = \mathbb{C}^n / \Lambda$, where $\Lambda \subseteq \mathbb{C}^n$ is rank-$2n$ sublattice lattice $+$ an ample polarization.

Example

X an **elliptic curve**. In $\mathbb{P}^2$

$$x_0^3 + x_1^3 + x_2^3 = 0.$$

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- (Product of) Calabi–Yau varieties;

Definition

X simply conn. trivial canonical bundle, $H^i(X, \mathcal{O}_X) = 0$, for $0 < i < \dim(X)$.

Example

X the quintic 3-fold.

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Assume X smooth proj. with $c_1 = 0$. Up to a finite étale map, X is isomorphic to a product varieties of the following types:

- Abelian variety;
- (Product of) Calabi–Yau varieties;
- (Product of) Irreducible holomorphic symplectic manifolds.

Definition

X simply connected $+$ trivial canonical bundle $+$ $H^2(X, \mathcal{O}_X) \cong \mathbb{C}$ generated by an everywhere non-deg. holomorphic 2-form.

Example

$\mathrm{Hilb}^n(\mathrm{K3})$ $=$ Hilbert scheme of length-n 0-dim. subschemes of a K3 surface.

The results

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More results:

- Additional results on abelian 3-folds by Maciocia–Piyaratne.
- More Calabi–Yau 3-folds: Bayer–Macrì–S., Koseki,...

The results

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Let $n \geq 2$ be an integer. Let X be a *very general* member of one of the following families

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Here, at the moment:

'very general'=infinite dense set containing inf. many very gen. examples in class. sense.

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The case of abelian n-fods answers a question of Pandharipande.

Ideas from the proof & future applicarions

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Table of Contents

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$$\mathrm{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle,$$

where:

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Enriques surfaces

X smooth projective surface $H^1(X, \mathcal{O}_X) = 0$ and $2K_X \equiv 0$.

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- **BLMS+Zhao, Li–Pertusi–Zhao:** For the special stab. cond. in the theorem above: $F(X) \cong M_\sigma(X)$=special moduli space of σ-stable objects in $\mathcal{K}u(X)$ (with Bayer–Macrì ample polarization).
  The isomorphism preserve special polarizations.

- Let $\varphi\colon H^4(X_1,\mathbb{Z}) \cong H^4(X_2,\mathbb{Z})$ be a Hodge isometry preserving the special classes H_1^2 and H_2^2 (H_i the hyperplane section).

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Then we reproved:

Torelli Theorem for cubic 4-folds (Voisin, Invent. Math., 1986)

Let X_1 and X_2 be cubic 4-folds. Then $X_1 \cong X_2$ iff there is a Hodge iso $H^4(X_1,\mathbb{Z}) \cong H^4(X_2,\mathbb{Z})$ preserving H_i^2 .