Geometry of Schemes

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Exercise Sheet

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Exercise 1.1 Let $\varphi\colon A \to B$ be an integral homomorphism of commutative rings. Show that $\mathrm{Spec}(\varphi)$ maps a closed point to a closed point and that the preimage of a closed point is a closed point.

Exercise 1.2 Let $\mathcal{B}$ be a base of open subsets on a topological space X (with this we mean that $\mathcal{B}$ is closed under finite intersections). Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves (of abelian groups) on X. Suppose that, for any $U \in \mathcal{B}$ there exists a homomorphism $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with restrictions. Show that this extends in a unique way to a morphism of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$.

Exercise 1.3 Let A be a graded ring and let Z be a reduced closed subscheme of $\mathrm{Proj}(A)$. Show that there exists a homogeneous ideal I of A such that $Z \cong \mathrm{Proj}(A/I)$.

Exercise 1.4 Show that an open subscheme of an algebraic variety over a field is an algebraic variety as well.

Exercise 1.5 Show that an open immersion is of finite type if it is quasi-compact. Show that in a scheme which is of finite type over a field, the set of closed points is dense.

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