arXiv:math.AG/0210465 v1 30 Oct 2002

THE CHOW GROUP OF THE MODULI SPACE OF MARKED CUBIC SURFACES

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ABSTRACT. Naruki gave an explicit construction of the moduli space of marked cubic surfaces, starting from a toric variety and proceeding with blow ups and contractions. Using his result, we compute the Chow groups and the Chern classes of this moduli space. As an application we relate a recent result of Freitag on the Hilbert polynomial of a certain ring of modular forms to the Riemann-Roch theorem for the moduli space.

Dedicated to the memory of our friend Fabio Bardelli

Following on the work of Allcock, Carlson and Toledo [ACT], which identified the moduli space of marked cubic surfaces $\mathcal{M}$ as a ball quotient, there has been a renewed interest in moduli spaces of cubic surfaces. In particular, Allcock and Freitag [AF] found a projective embedding of $\mathcal{M}$ using new results of Borcherds on modular forms. This map was actually described earlier by Coble, who identified $\mathcal{M}$ with the moduli space of six points in $\mathbf{P}^2$. We will thus call this map the CAF-map.

The moduli space $\mathcal{M}$ is smooth except for 40 singular points, the cusps. Blowing up the cusps, one obtains a smooth projective variety $\mathcal{C}$ which we refer to as Naruki's cross ratio variety. Using basic work of Cayley on cubic surfaces and associated projective invariants, certain cross ratios, Naruki [N] realized that $\mathcal{C}$ could be obtained from a toric variety associated to the root system D_4 via a process of explicit blow ups and contractions. Moreover he showed that there is a biregular action of $W(E_6)$ on $\mathcal{C}$. His description of $\mathcal{C}$ is at the basis of this paper.

We determine the Chow groups of $\mathcal{C}$ and the spaces of $W(E_6)$-invariant cycles in section 2. In section 3 we consider the exceptional divisors in $\mathcal{C}$ over the cusps in $\mathcal{M}$, which we call cusp divisors (these are actually very simple varieties, being the product of three $\mathbf{P}^1$'s). We study the 36 boundary divisors, which parametrize nodal cubic surfaces, in section 4. The image of such a divisor in $\mathcal{M}$ is isomorphic to the Segre cubic in $\mathbf{P}^4$. We use the information obtained on these divisors to compute the Chern classes of $\mathcal{C}$ in section 5. In section 6.3 we consider the 45 tritangent divisors, these parametrize cubics with an Eckardt point (a point on the cubic surface through which 3 lines pass). All these divisors correspond, in a $W(E_6)$-equivariant way, to points in a finite projective geometry. For the convenience of the reader we added some tables at the end of the paper which give such a correspondence.

The main application of our results is an explicit form of the Riemann-Roch theorem on the fourfold $\mathcal{C}$ in section 5. It allows us to compute the Euler characteristic of integral multiples of the divisor class which gives the CAF-map. The quartic polynomial we found (see Theorem 6.2) agrees with the Hilbert function of a certain graded ring of modular forms associated to the CAF-map which was recently determined by Freitag [Fr]. We also obtain some results on the Picard group of the moduli space of smooth marked cubic surfaces in section 2.5.

Finally, we should probably point out that in using Naruki's description, we do not (need to) consider families of cubic surfaces and the 'tautological' classes associated to them. This is somewhat unfortunate, as this might give more insight in the structure of the Chow groups of $\mathcal{C}$ viewed as compactification of a moduli space.

1. The cross ratio variety

1.1. **The toric variety $\tilde{T}$.** The starting point of Naruki's construction is the smooth toric variety $T \cong (\mathbf{C}^*)^4 \hookrightarrow \tilde{T}$ defined by the fan in the weight lattice $N \cong \mathbf{Z}^4$ of the root system of type D_4 whose 4-dimensional cones are the 192 Weyl chambers, see [N] and [vG], section 2. This fan has 48 edges (i.e. 1-dimensional cones), which correspond to the divisors in the boundary of $\tilde{T}$. Identifying $N \otimes_{\mathbf{Z}} \mathbf{R}$ with $\mathbf{R}^4$, with standard basis $\epsilon_1, \dots, \epsilon_4$ and standard scalar product, let

$$S := \{\pm\epsilon_i\} \cup \{(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2\}, \qquad R := \{\pm\epsilon_i \pm \epsilon_j\},$$

then S and R each have 24 elements and the half-lines they generate are the edges of the fan. The elements from $S \cup R$ generate the $\mathbf{Z}$-module N, for $\tau \in S \cup R$ we denote by $V(\tau)$ $(\subset \tilde{T})$ the corresponding divisor.

The character group of T is identified with $M := Hom(N, \mathbf{Z})$, the root lattice of D_4. A $\mathbf{Z}$-basis of M is

$$e_1 - e_2, \quad e_2 - e_3, \quad e_3 - e_4, \quad e_3 + e_4,$$

where the e_i are the dual basis of the ϵ_i, the corresponding characters $T \to \mathbf{C}^*$ are usually denoted by λ, ρ, ν and μ respectively.

1.2. **The cross ratio variety $\mathcal{C}$.** Naruki's (smooth, projective) cross ratio variety $\mathcal{C}$ is obtained from the toric variety $\tilde{T}$ as follows ([N], §10-12):

$$\mathcal{C} \stackrel{r}{\longleftarrow} \hat{T} \stackrel{\pi''}{\longrightarrow} \tilde{T}'' \stackrel{\pi'}{\longrightarrow} \tilde{T}' \stackrel{\pi_e}{\longrightarrow} \tilde{T}.$$

The map π_e is the blow up of $\tilde{T}$ in the identity element $e \in T$. The exceptional divisor $\pi_e^{-1}(e) \cong \mathbf{P}^3$ is denoted by $\mathbf{P}^3_{\mathrm{w}}$. The image in $\mathcal{C}$ of its strict transform $\hat{E}$ in $\hat{T}$ is a tritangent divisor denoted by $D_{\mathrm{w}} = D_{(16)}$.

The map π' is the blow up of $\tilde{T}'$ in 12 smooth, disjoint, rational curves $C'_1, \dots, C'_{12}$. We write $C''_j := (\pi')^{-1}(C'_j)$ for the exceptional divisor, the map $C''_j \to C'_j$ induced by π' is a $\mathbf{P}^2$-bundle.

The map π'' is the blow up in 16 smooth disjoint surfaces $S''_1, \dots, S''_{16} \subset \tilde{T}''$. Each exceptional divisor C''_j meets 4 of the surfaces S''_i in 4 disjoint rational curves. These 4 curves are sections of the $\mathbf{P}^2$-bundle $C''_j \to C'_j$ which meet each fiber of the $\mathbf{P}^2$-bundle $C''_j \to C'_j$ in 4 general points (cf. [N], Prop. 9.1, §10). The strict transforms $\hat{C}_j$ in $\hat{T}$ of the C''_j are trivial bundles:

$$\hat{C}_j \cong \mathbf{P}^1 \times V \stackrel{\pi' \circ \pi''}{\longrightarrow} \mathbf{P}^1 \cong C'_j$$

here V is the blow up of $\mathbf{P}^2$ in 4 points, the map induced by $\pi' \circ \pi''$ is the projection on the first factor. The map $r : \hat{T} \to \mathcal{C}$ induces the projection $\hat{C}_j \to V$ on the second factor and r is an isomorphism on the complement of the 12 $\hat{C}_i$'s ([N], p. 22, 23 and Prop. 11.3).

The 16 exceptional divisors $\hat{S}_i$ of π'' in $\hat{T}$ map under r to divisors in $\mathcal{C}$. The $W(E_6)$-orbit of these 16 divisors in $\mathcal{C}$ consists of 40 divisors, the other 24 are the images under r of the strict transforms of the $V(\tau)$'s with $\tau \in R$ ([N], Prop. 11.2). We call these 40 divisors the cusp divisors of $\mathcal{C}$.

1.3. **The marked moduli space $\mathcal{M}$.** There is a morphism

$$c : \mathcal{C} \longrightarrow \mathcal{M},$$

where $\mathcal{M}$ is the moduli space of semistable marked cubic surfaces, which contracts the 40 cusp divisors to (singular) points (cf. [N], Introduction and §12), the cusps of $\mathcal{M}$. The map c is an isomorphism between the complement of the cusp divisors in $\mathcal{C}$ and the complement of the 40 cusps of $\mathcal{M}$. The Weyl group $W(E_6)$ acts biregularly on both $\mathcal{C}$ and $\mathcal{M}$, the morphism c is $W(E_6)$-equivariant.

There is a $W(E_6)$-equivariant embedding, the CAF-map

$$F : \mathcal{M} \longrightarrow \mathbf{P}^9$$

which was studied extensively by Coble, Allcock and Freitag [AF], [Fr2] and also in [vG].

1.4. **The cusp divisors.** The cross ratio variety $\mathcal{C}$ has 40 cusp divisors of $\mathcal{C}$, each of these is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, cf. [N], §12.

1.5. **The boundary divisors.** A boundary divisor is an irreducible component of the closure of the set of marked cubics with a node. There are 36 such divisors in $\mathcal{C}$, 24 of which are obtained from the $V(\tau)$ with $\tau \in S$, the other 12 come from the subtori defined by $\alpha = 1$ in T where α is a positive root of D_4.

1.6. **The tritangent divisors.** A tritangent divisor is an irreducible component of the closure of the set of marked cubics with an Eckardt point, that is a point on a cubic surface where three lines meet. There are 45 such divisors in $\mathcal{C}$. One of them is obtained from $\mathbf{P}^3_{\mathrm{w}}$, the other 44 are defined by the explicit equations in [N], Table 3.

1.7. **Incidence between divisors.** The cusp, boundary and tritangent divisors can be very conveniently parametrized by points in a finite projective orthogonal geometry. Let

$$b : \mathbf{F}_3^5 \times \mathbf{F}_3^5 \longrightarrow \mathbf{F}_3, \qquad b(x,y) = \sum_i x_i y_i$$

be the standard bilinear form on $\mathbf{F}_3^5$ and let $q(x) = b(x,x)$ be the associated quadratic form. The orthogonal group $O(\mathbf{F}_3^5, q)$ is isomorphic to $W(E_6) \times \{\pm 1\}$. Note that in [AF] the quadratic form $x_1^2 - x_2^2 - \ldots - x_5^2$ is used, substituting $x_2 := x_2 + x_3$, $x_3 = x_2 - x_3$ and similarly for x_4, x_5, one obtains $q(x)$ since $-2 \equiv 1 \bmod 3$.

As the nonzero elements in $\mathbf{F}_3$ are ± 1, the subsets

$$N_i := \{ z \in \mathbf{P}(\mathbf{F}_3^5) : \ q(z) = i \}$$

for $i \in \mathbf{F}_3$ are well defined. It is easy to check that

$$\sharp N_0 = 40, \qquad \sharp N_1 = 45, \qquad \sharp N_{-1} = 36.$$

The group $W(E_6)$ acts transitively on each of these sets.

There is a natural $W(E_6)$-equivariant identification between N_0 and the set of cusps, N_1 and the set of tritangent divisors, N_{-1} and the set of boundary divisors, see [AF], section 3.

There it is also shown that a cusp divisor and a boundary (tritangent) divisor have non-empty intersection iff the corresponding lines in $\mathbf{F}_3^5$ are perpendicular. In particular, a boundary divisor meets 10 and a tritangent divisor meets 16 cusp divisors.

Lemma 3.1 of [AF] shows that if D_z and D_w are boundary divisors corresponding to $z, w \in N_{-1}$ then $D_z \cap D_w$ is non-empty iff the lines $z, w \subset \mathbf{F}_3^5$ are perpendicular. This implies that a boundary divisor meets 15 other boundary divisors.

A tritangent divisor has non-empty intersection with any other tritangent divisor ([vG], 6.7) and with any boundary divisor (for this it is most convenient to use the description of these divisors as images of long and short mirrors in the 4-ball from [AF], section 3).

If $v \in \mathbf{F}_3^5$ is a vector with $q(v) \neq 0$, the map $s_v : x \mapsto x + q(v)b(x,v)v$ is in $O(\mathbf{F}_3^5, q)$, note that $s_v = s_{-v}$. For $v \in N_{-1}$ the map s_v corresponds to the reflection in a roots of E_6. For $v \in N_1$, the map $-s_v$ corresponds to an involution in $W(E_6)$. The divisor D_v, for $v \in N_{-1} \cup N_1$, is the unique divisor in the fixed point set of the involution $\pm s_v$ in $\mathcal{C}$.

2. The Chow groups

2.1. For a smooth variety X we denote by $A^i(X)$ the Chow group of codimension i-cycles modulo rational equivalence. For a ring R we write $A^i(X)_R := A^i(X) \otimes_{\mathbf{Z}} R$. We determine the Chow groups of $\mathcal{C}$ in Theorem 2.4, some results on $A^2(\mathcal{C})$ are actually proved in 4.16. Various intersection numbers are computed in 4.13 and 3.2.

2.2. There are three divisors on $\mathcal{C}$ which are obviously invariant under $W(E_6)$, they are:

$$\hat{B}, \qquad \hat{C}, \qquad \hat{T}$$

which are the sum of the 36 boundary divisors, the sum of the 40 cusp divisors and the sum of the 45 tritangent divisors (actually we already used the notation $\hat{T}$ in 1.2, however no confusion is possible). Theorem 2.4.2 shows that these classes are linearly dependent in $A^1(\mathcal{C})$.

2.3. **Symmetrizing.** Let D be a divisor on $\mathcal{C}$ which is a linear combination of tritangent, boundary and cusp divisors. If the class of D in the Chow group $A^1(\mathcal{C})$ is invariant, then σ^*D has the same class for any $\sigma \in W(E_6)$. Therefore the class of D is the same as the class of the sum over $\sigma \in W(E_6)$ of the σ^*D divided by $\sharp W(E_6)$.

Since $W(E_6)$ permutes the 45 tritangent divisors transitively (and similarly the 36 boundary divisors as well as the 40 cusp divisors) the net result is that in D we replace each tritangent divisor by $\frac{1}{45}\hat{T}$ etc.

2.4. **Theorem.**

(1) The Chow groups of the cross ratio variety $\mathcal{C}$ are free $\mathbf{Z}$-modules of rank:

$$\mathrm{rk}\, A^0(\mathcal{C}) = \mathrm{rk}\, A^4(\mathcal{C}) = 1, \qquad \mathrm{rk}\, A^1(\mathcal{C}) = \mathrm{rk}\, A^3(\mathcal{C}) = 61, \qquad \mathrm{rk}\, A^2(\mathcal{C}) = 147.$$

(2) The subgroups of $W(E_6)$-invariant classes have rank:

$$\mathrm{rk}\, A^i(\mathcal{C})^{W(E_6)} = 1, \qquad \mathrm{rk}\, A^j(\mathcal{C})^{W(E_6)} = 2 \qquad (i = 0, 4, \quad j = 1, 2, 3).$$

The classes $\hat{B}^i$, $\hat{C}^i$ are a basis of $A^i(\mathcal{C})_{\mathbf{Q}}^{W(E_6)}$, we have:

$$\hat{T} = (25\hat{B} + 27\hat{C})/4 \in A^1(\mathcal{C}), \qquad \hat{B}\hat{C} = -3\hat{C}^2 \in A^2(\mathcal{C}).$$

(3) The canonical class of the cross ratio variety is:
$$K_{\mathcal{C}} = (-\hat{B} + \hat{C})/4.$$
(4) The decomposition of $A^1(\mathcal{C})$ and $A^3(\mathcal{C})$ into irreducible $W(E_6)$-representations is:
$$A^1(\mathcal{C})_{\mathbf{C}} \cong A^3(\mathcal{C})_{\mathbf{C}} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{15_q} \oplus \mathbf{20_p} \oplus \mathbf{24_p}.$$
(5) The cusp divisors are linearly independent in $A^1(\mathcal{C})$, the 40-dimensional subspace of $A^1(\mathcal{C}_{\mathbf{C}})$ which they span decomposes as:
$$\mathbf{1} \oplus \mathbf{15_q} \oplus \mathbf{24_p}.$$
Similarly, the boundary divisors span a 36-dimensional subspace of $A^1(\mathcal{C})_{\mathbf{C}}$ which decomposes as:
$$\mathbf{1} \oplus \mathbf{15_q} \oplus \mathbf{20_p},$$
and the tritangent divisors span a 45-dimensional subspace:
$$\mathbf{1} \oplus \mathbf{20_p} \oplus \mathbf{24_p}.$$

Proof. The construction of $\mathcal{C}$ via the diagram in 1.2 shows that the Chow groups of $\mathcal{C}$ are free $\mathbf{Z}$-modules of finite rank and that $A^i(\mathcal{C}) \cong H^{2i}(\mathcal{C}, \mathbf{Z})$ whereas $H^{2i+1}(\mathcal{C}, \mathbf{Z}) = 0$. Since all varieties in the diagram in 1.2, are smooth, so $A^1 \cong A^3$, we concentrate on A^1 and A^2.

The rank of $A^1(\tilde{T})$ is given by the number of edges in the fan defining $\tilde{T}$, which is 48, minus the rank of the torus, which is 4. The Euler characteristic of $\tilde{T}$ is equal to the number of maximal cones ([Fu2], p.59), which is 192. Hence the rank of A^2 is $192 - 2 - 2 \cdot 44$:
$$\operatorname{rk} A^1(\tilde{T}) = 44, \qquad \operatorname{rk} A^2(\tilde{T}) = 102.$$

The Chow groups of the blow up $\tilde{T}'$ of $\tilde{T}$ in the identity element are $A^i(\tilde{T}') \cong A^i(\tilde{T}) \oplus A^i(\mathbf{P}^3_{\mathrm{w}})$, for $i = 1, 2$ hence the ranks increase by 1.

The variety $\tilde{T}''$ is obtained from $\tilde{T}'$ by blowing up the 12 disjoint C_j''s, each of which is a $\mathbf{P}^1$, hence we get:
$$\operatorname{rk} A^1(\tilde{T}'') = 45 + 12 = 57, \qquad \operatorname{rk} A^2(\tilde{T}'') = 103 + 24 = 127.$$

Next one blows up 16 surfaces S_i'' in $\tilde{T}''$ to obtain the variety $\hat{T}$. These surfaces are the strict transforms in $\tilde{T}''$ of surfaces $S_i \subset \tilde{T}$. Each S_i is the closure of a subtorus (for example $\nu = \rho = 1$ defines an S_i, the first table in 7.6 lists all 16 surfaces). Using the description of the fan defining $\tilde{T}$ one finds that $\operatorname{rk}A^1(S_i) = 4$ for all i. These surfaces all contain $e \in \tilde{T}$ hence $\operatorname{rk}A^1(S_i') = 5$, where S_i' is the strict transform of S_i in $\tilde{T}'$. Each C_j' is the strict transform of the closure C_j of a 1-dimensional subtorus of $\tilde{T}$ ([N], p.20), hence C_j is either contained in S_i or meets it in e. As e gets blown up, an S_i' either contains a C_j' or they are disjoint. Thus $\pi'' : S_i'' \to S_i'$ is an isomorphism and $\operatorname{rk}A^1(S_i'') = 5$. Therefore:
$$\operatorname{rk} A^1(\hat{T}) = 57 + 16 = 73, \qquad \operatorname{rk} A^2(\hat{T}) = 127 + 5 \cdot 16 = 207.$$

Finally we consider the blow down $r : \hat{T} \to \mathcal{C}$, it contracts the 12 $\hat{C}_j \cong \mathbf{P}^1 \times V$ in $\hat{T}$ to the surfaces V in $\mathcal{C}$. Since V is a $\mathbf{P}^2$ blown up in 4 points we get:
$$\operatorname{rk} A^1(\mathcal{C}) = 73 - 12 = 61, \qquad \operatorname{rk} A^2(\mathcal{C}) = 207 - 5 \cdot 12 = 147.$$

A relation between the invariant divisor classes can be obtained from the divisor of the rational function $\lambda - 1$ on $\mathcal{C}$, see section 7.2. This divisor can be written as $(\lambda-1)_0 - (\lambda-1)_\infty$ where $(\lambda-1)_0$ is the sum of one tritangent, one boundary and 4 cusp divisors, whereas $(\lambda-1)_\infty$ is the sum of 6 boundary and 9 cups divisors, one of which has multiplicity two. Symmetrizing this relation in $A^1(\mathcal{C})$, we get:

$$0 = \frac{1}{45}\hat{T} + \frac{1}{36}(1-6)\hat{B} + \frac{1}{40}(4-10)\hat{C},$$

from which we get the relation $4\hat{T} = 25\hat{B} + 27\hat{C}$ stated in the theorem. The relation in A^2 will be proved in Proposition 3.2 by restriction to a cusp divisor.

The canonical divisor of the toric variety $\tilde{T}$ is minus the sum of the 48 divisors $V(\tau)$ ([Fu2], §4.3), $\tau \in R \cup S$, half of which give boundary divisors, the others give cups divisors. Blowing up the origin, we must add $3\mathbf{P}^3_{\mathrm{w}}$ to the strict transform of $K_{\tilde{T}}$ to obtain $K_{\tilde{T}'}$ ([GH], Lemma in Chap. 1.4, p.187). The exceptional divisors for π' are contracted by r, so these do not contribute, however we must add the 16 cups divisors which come from the blow up of the surfaces in $\tilde{T}''$. Symmetrizing the result we get:

$$K_C = \frac{3}{45}\hat{T} + \frac{-24}{36}\hat{B} + \frac{1}{40}(-24+16)\hat{C} = (-\hat{B}+\hat{C})/4$$

where we used the relation obtained before to eliminate $\hat{T}$. A nice, explicit, anti-canonical divisor is given at the end of 7.1.

From the construction of $\mathcal{C}$ it is clear that $A^1(\mathcal{C})$ is generated by the tritangent, boundary and cusp divisors. The divisor of $(\lambda-1)$ on $\mathcal{C}$ contains exactly one tritangent divisor with multiplicity one and this shows that the 36 boundary divisors and the 40 cusp divisors suffice to generate $A^1(\mathcal{C})$. Since $\mathrm{rk}A^1(\mathcal{C})$=61, there must be 15 independent relations between the boundary and cusp divisors. One such relation, which involves both boundary and cusp divisors, is given by the divisor of the rational function λ (see 7.1), others can be obtained by applying $W(E_6)$ to this relation.

The group $W(E_6)$ permutes the boundary and the cusp divisors, the decomposition of these permutation representations can be found in the atlas [At], p. [26], $U_4(2)$, see also [F] whose notation we use. The permutation representations on the 36 boundary components and the 45 tritangents are denoted by $\chi^{(36)}$ and $\chi^{(45)}$ in [F], p. 100. The group $W(E_6)$ has two permutation representations of degree 40 (see [At]), the one which corresponds to the 40 cusp divisors is not $\chi^{(40)}$ in [F] but it is the one listed in the theorem, as can be verified by computing a few traces.

The only 15-dimensional $W(E_6)$ representation which the two permutation representations have in common is the irreducible representation $\mathbf{15_q}$, hence this must be the representation on the relations. Therefore the representation on $A^1(\mathcal{C})$ is the sum of the two permutation representations modulo $\mathbf{15_q}$. In particular $A^1(\mathcal{C})^{W(E_6)}$ has rank 2 and is generated (over $\mathbf{Q}$) by $\hat{B}$ and $\hat{C}$. Since the permutation representation on the tritangents does not contain $\mathbf{15_q}$ and since the relation provided by $\lambda-1$ involves both boundary and cusp divisors, the subrepresentations $\mathbf{20_p}$ (which is in common with the boundary divisors) and $\mathbf{24_p}$ (which is in common with the cusps) cannot map to zero in $A^1(\mathcal{C})$, and finally since $\hat{T} \in A^1(\mathcal{C})$ is non-zero as well, we conclude that the tritangents divisors are independent in $A^1(\mathcal{C})$.

The proof of $\mathrm{rk}A^2(\mathcal{C})^{W(E_6)} = 2$ will be given in 4.16. □

2.5. **Remark.** Let $\mathcal{M}^0 \subset \mathcal{M}$ be the moduli space of smooth marked cubic surfaces, so $\mathcal{M}^0 \cong \mathcal{C} - \mathrm{Support}(\hat{B} + \hat{C})$, and it is also the complement in $\mathcal{M}$ of the 36 boundary divisors. As we observed in the proof of the theorem, the first Chow group of $\mathcal{C}$ is generated by the classes of the boundary and cusp divisors. In particular,

$$A^1(\mathcal{M}^0) = Pic(\mathcal{M}^0) = 0.$$

(Note that relations $\hat{T} = (25\hat{B} + 27\hat{C})/4$ and $K_{\mathcal{C}} = (-\hat{B} + \mathcal{C})/4$ only show that $4\hat{T}$ and $4K_{\mathcal{C}}$ restrict to zero on $\mathcal{M}^0$). Since finite cyclic subgroups of $Pic(\mathcal{M}^0)$ correspond to finite cyclic unramified coverings of $\mathcal{M}^0$, we conclude that the maximal abelian quotient of $\pi_1(\mathcal{M}^0)$ is trivial. See [L] for interesting results on this fundamental group.

2.6. **The hyperplane class.** We determine $(F \circ c)^*\mathcal{O}(1)$ where (cf. section 1.3)

$$\mathcal{C} \xrightarrow{c} \mathcal{M} \xrightarrow{F} \mathbf{P}^9.$$

From the work of Allcock and Freitag [AF] we know that, modulo cusp divisors, it is given by a so called cross divisor, that is

$$(F \circ c)^*\mathcal{O}(1) \equiv D_\alpha + D_\beta + D_\gamma + D_\delta + D_t \qquad \text{(modulo cusp divisors)},$$

where $\alpha, \dots, \delta \in N_{-1}$ and $t \in N_1$ are pairwise perpendicular. In particular, $D_\alpha, \dots, D_\delta$ are boundary divisors and D_t is a tritangent divisor.

The divisor class $(F \circ c)^*\mathcal{O}(1) \in A^1(\mathcal{C})$ is $W(E_6)$-invariant (since $F \circ c$ is $W(E_6)$-equivariant), and its expression in terms of invariant classes is given in the following proposition.

2.7. **Proposition.** The $W(E_6)$-invariant hyperplane class of the CAF map is given by:

$$(F \circ c)^*\mathcal{O}(1) = (\hat{B} + 3\hat{C})/4.$$

Proof. The contribution of the cusp divisors to $D_\alpha + D_\beta + D_\gamma + D_\delta + D_t$ is the sum of the 24 cusp divisors D_n, each with multiplicity one, where $n \in N_0$ is perpendicular to at least one of element of the set $\{\alpha, \beta, \gamma, \delta, t\}$. This can be seen from a local computation using the explicit expression of the coordinate functions of $(F \circ c)$ (cf. [vG], 5.5 and the proof of Theorem 5.7).

Symmetrizing w.r.t. $W(E_6)$, as explained in 2.3, we obtain the result. □

3. Cusp divisors.

3.1. According to Naruki, [N], each cusp divisor is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. In the following proposition we prove the relation between $W(E_6)$-invariant codimension two classes (already stated in Theorem 2.4) and we determine some intersection numbers involving cusp divisors.

3.2. **Proposition.** Let $C_0 \cong (\mathbf{P}^1)^3$ be a cusp divisor, and let

$$D_1 = \{0\} \times (\mathbf{P}^1)^2, \qquad D_2 = \mathbf{P}^1 \times \{0\} \times \mathbf{P}^1, \qquad D_3 = (\mathbf{P}^1)^2 \times \{0\} \qquad (\in A^1(C_0)).$$

There are nine boundary divisors which have non-empty intersection with C_0. The class of the intersection of each of these with C_0 is one of the D_i and

$$C_0\hat{B} = 3(D_1 + D_2 + D_3).$$

Let B_i be a boundary divisor such that $B_iC_0 = D_i$. Then, for $i \neq j$:

$$C_0B_1B_2B_3 = 1, \qquad C_0B_i^2B_j = 0, \qquad C_0B_i^3 = 0, \qquad C_0^2B_iB_j = -1, \qquad C_0^2B_i^2 = 0.$$

Moreover,
$$C_0^2 = -C_0(B_1 + B_2 + B_3), \qquad C_0^3 B_i = 2, \qquad C_0^4 = -6.$$
Finally we have:
$$\hat{B}\hat{C} = -3\hat{C}^2 \in A^2(\mathcal{C}), \qquad \hat{C}^4 = -240.$$

Proof. The toric subvariety $V(\epsilon_1 + \epsilon_3)$ in $\tilde{T}$ is defined by the fan consisting of the images in $\mathbf{R}^4/\langle\epsilon_1 + \epsilon_3\rangle$ of all the cones containing $\epsilon_1 + \epsilon_3$. One finds that this fan has six edges and that $V(\epsilon_1 + \epsilon_3) \cong (\mathbf{P}^1)^3$. The divisor $V(\epsilon_1 + \epsilon_3)$ is birationally isomorphic to a cusp divisor C_0 in $\mathcal{C}$ via the maps in the diagram in 1.2 above. One checks that each of these maps is an isomorphism on this divisor, hence C_0 is isomorphic to $(\mathbf{P}^1)^3$.

The six edges of the fan defining $C_0 = V(\epsilon_1+\epsilon_3)$ define the classes D_i (each twice) in $A^1(C_0)$. On the other hand, the edges correspond to non-empty intersections $V(\epsilon_1 + \epsilon_3) \cap V(\tau)$. One verifies that $\tau \in S$, hence the $V(\tau)$ define boundary divisors. In particular, each $V(\epsilon_1+\epsilon_3)\cap V(\tau)$ is a D_i. That there are nine boundary divisors meeting C_0 follows from 1.7. In the notation of 7.6, $V(\epsilon_1 + \epsilon_3)$ is labelled as [24.35.16]. The nine roots in this set correspond to the nine boundary divisors meeting C_0. These nine roots are in three orthogonal A_2's, the three divisors corresponding to the roots from one A_2 intersect C_0 in the same class in $A^1(C_0)$ because the corresponding divisors do not meet on $\mathcal{C}$. In particular $C_0\hat{B} = 3(D_1 + D_2 + D_3)$.

Now it follows that $C_0B_1B_2B_3 = 1$ because the D_i intersect transversely in one point on C_0, for $C_0^2B_iB_j$ see below. Finally $C_0B_i^2B_j = C_0B_i^3 = C_0^2B_i^2 = 0$ since $D_i^2 = 0$ in $(\mathbf{P}^1)^3$.

To find the selfintersection C_0^2, we restrict the divisor of the rational function λ (cf. 7.1) to $C_0 = V(\epsilon_1+\epsilon_3) = D_v$ with $v = (1,-1,0,-1,0) \in N_0$. Using 1.7 and the tables in the Appendix to eliminate empty intersections, we find: $C_0(C_0+B_1+B_2+B_3) = 0$ where the B_i are the $V(\tau)$ with $\tau = \epsilon_1$, $(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4)/2$,$(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)/2$, which correspond to the roots h_{345}, h_{136} and h_{246}. These three roots are in three distinct A_2's in [24.35.16] so $C_0^2 = -(D_1 + D_2 + D_3)$. Using this relation twice we get:
$$C_0^3B_i = C_0^2(-B_1 - B_2 - B_3)B_i = C_0(-B_1 - B_2 - B_3)^2B_i = 2.$$
Therefore $C_0^4 = C_0^2 \cdot C_0(-B_1 - B_2 - B_3) = -3 \cdot 2 = -6$. We also get $C_0^2B_iB_j = -(D_1 + D_2 + D_3)D_iD_j = -1$ if $i \neq j$.

Since $\hat{C}$ is the sum of 40 disjoint cusp divisors, it follows that $\hat{C}^4 = 40 \cdot -6 = -240$. Finally we observe that $C_0\hat{B} = 3(D_1 + D_2 + D_3) = -3C_0^2$, hence, summing over the 40 cusps, we get $\hat{C}\hat{B} = -3\hat{C}^2$. □

3.3. **Remark.** Recall from Theorem 2.4 that $rk(A^2(\mathcal{C})) = 147$. We identify 120 independent codimension two cycles.

Let $C_0 \cong (\mathbf{P}^1)^3$ be a cusp divisor and let $D_i = C_0B_i \in C_0$ be the three divisors on C_0 as in Proposition 3.2. Then $D_iD_j = -1$, if $i \neq j$, and $D_i^2 = 0$ in $A^4(\mathcal{C})$. Therefore the D_i are independent in $A^2(\mathcal{C})$. As the 40 cusps are disjoint we get $3\cdot 40 = 120$ independent codimension two classes on $\mathcal{C}$.

4. The boundary divisors

4.1. In this section we study the boundary divisors and we compute some intersection numbers on $\mathcal{C}$ in Theorem 4.13.

4.2. **The Chow group of a boundary divisor.** Let B_0 be a boundary divisor on $\mathcal{C}$. Recall that the symmetric group S_6 acts on B_0. The divisor B_0 meets 15 other boundary divisors, 10 cusp divisors and the 45 tritangent divisors. We denote the sum of these by

$$\hat{B}_b, \qquad \hat{C}_b, \qquad \hat{T}_b.$$

Then in $A^1(B_0)$ we have:

$$\hat{B}_{|B_0} = \hat{B}_b + B_0^2, \qquad \hat{C}_{|B_0} = \hat{C}_b, \qquad \hat{T}_{|B_0} = \hat{T}_b.$$

4.3. **Proposition.** The Chow groups of a boundary divisor B_0 are:

$$A^1(B_0) \cong A^2(B_0) \cong \mathbf{Z}^{16}.$$

We also have:

$$A^1(B_0)^{S_6} \cong A^2(B_0)^{S_6} \cong \mathbf{Z}^2,$$

$\hat{B}_b^i$, $\hat{C}_b^i$ are a basis of $A^i(B_0)^{S_6}_{\mathbf{Q}}$ for $i = 1, 2$ and $\hat{B}_b\hat{C}_b = -3\hat{C}_b^2$. The intersection numbers between invariant classes are:

$$\hat{B}_b^3 = -165 = -3 \cdot 5 \cdot 11, \qquad \hat{B}_b^2\hat{C}_b = 180 = 2^2 \cdot 3^2 \cdot 5,$$
$$\hat{B}_b\hat{C}_b^2 = -60 = -2^2 \cdot 3 \cdot 5, \qquad \hat{C}_b^3 = 20 = 2^2 \cdot 5.$$

Moreover we have:

$$c_1(N_{B_0/\mathcal{C}}) = B_0^2 = -(\hat{B}_b + 3\hat{C}_b)/5, \qquad K_{B_0} = -(2\hat{B}_b + \hat{C}_b)/5.$$

Proof. To compute the Chow groups of a boundary divisor, we choose B_0 to be the strict transform of the divisor $V(\epsilon_1)$ on the toric variety $\tilde{T}$. The divisor $V(\epsilon_1)$ is the toric variety defined by the fan consisting of the images in $\mathbf{R}^4/ < \epsilon_1 >$ of all the cones which contain ϵ_1. One finds that $V(\epsilon_1)$ is smooth and that the fan has 14 edges. Thus $\mathrm{rk} A^1 V(\epsilon_1) = 14 - 3 = 11$. This divisor does not contain $e \in \tilde{T}$, but it meets (transversally in one point) exactly one of the 12 curves which get blown up in $\tilde{T}'$, it is $C_1' : \ \mu = \nu = \rho = 1$, hence $\mathrm{rk} A^1(V(\epsilon_1)'') = 12$. The divisor $V(\epsilon_1)''$ in $\tilde{T}''$ meets exactly 4 of the 16 surfaces which get blown up, it meets of them in a smooth rational curve. Therefore $\mathrm{rk} A^1 \widehat{V(\epsilon_1)} = 12 + 4 = 16$. Since $\widehat{V(\epsilon_1)}$ intersects just one of the $\hat{C}_i \cong \mathbf{P}^1 \times S$ (it is $\hat{C}_1$) in a copy of V ([N], p. 22) and r contracts $\hat{C}_1$ to V, we conclude that the blow down map $r : \hat{T} \to \mathcal{C}$ induces an isomorphism of smooth 3-folds $\widehat{V(\epsilon_1)} \cong B_0$. Hence $A^1(B_0)$ and $A^2(B_0)$ have rank 16.

Note that $K_{V(\varepsilon_1)}$ is minus the sum of the divisors corresponding to the 14 rays of the fan (the strict transform of 8 of these divisors is a boundary divisor, the other 6 are cusp divisors). From the formula for the canonical divisor of a blow up we find that $K_{V(\epsilon_1)''}$ is obtained from the strict transform of $K_{V(\varepsilon_1)}$ by adding twice the class of the (irreducible) exceptional divisor, this exceptional divisor is a boundary divisor. Finally $K_{\widehat{V(\epsilon_1)}} = K_{B_0}$ is obtained by adding 4 cusp divisors to this. Since K_{B_0} is S_6-invariant, we can symmetrize w.r.t. S_6:

$$K_{B_0} = \frac{1}{15}(-8+2)\widehat{B}_b + \frac{1}{10}(-6+4)\widehat{C}_b = -(2\hat{B}_b + \hat{C}_b)/5.$$

The selfintersection B_0^2 can be computed from the restriction of the divisor of λ on $\mathcal{C}$ (cf. 7.1) to B_0. As $B_0 = r_* \widehat{V(\epsilon_1)}$ occurs in the divisor of zeroes with multiplicity one, the relation $0 = B_0 \cdot (\lambda)$ implies that B_0^2 is a linear combination of 4 boundary divisors in B_0 with coefficient

-1 and 1 boundary divisor with coefficient 1, as well as 4 cusp divisors in B_0 with coefficient -1 and 1 with coefficient -2. Symmetrizing w.r.t. S_6 we get:

$$B_0^2 = -\frac{1}{15}(4-1)\widehat{B}_b - \frac{1}{10}(4+2)\widehat{C}_b = -(\hat{B}_b + 3\hat{C}_b)/5.$$

It is amusing to check that these expressions for $K_{\mathcal{C}}$, K_{B_0} and B_0^2 do indeed satisfy the adjunction formula:

$$K_{B_0} = (K_{\mathcal{C}} + B_0)_{|B_0}.$$

See 4.8 for the S_6-invariant classes and 4.11 for the intersection numbers. □

4.4. **The Segre cubic.** The Segre cubic is the cubic threefold S in $\mathbf{P}^5$ defined by the equations:

$$S: \qquad x_1 + \ldots + x_6 = x_1^3 + \ldots + x_6^3 = 0.$$

It is well know that any boundary divisor is isomorphic to the blow up of S in its 10 nodes (the points with three coordinates $+1$ and three coordinates -1). In fact, the image of B_0 in $\mathcal{M} \subset \mathbf{P}^9$ is the Segre cubic (cf. [vG], 6.2). The 10 nodes are the images of the intersection of B_0 with 10 cusp divisors, each such intersection is isomorphic to the exceptional divisor over a node, thus to $\mathbf{P}^1 \times \mathbf{P}^1$. The Segre cubic is obviously invariant under permutations of the x_i, this action of S_6 is induced by the action of a subgroup in $W(E_6)$ consisting of all the elements which fix B_0.

4.5. **Boundary surfaces in** S**.** The intersection of B_0 with a boundary divisor B_1 is, if non-empty, isomorphic to V, a $\mathbf{P}^2$ blown up in 4 general points. Thus V maps to subvariety isomorphic to $\mathbf{P}^2$ in S and this subvariety contains 4 of the singular points.

The divisor B_1 is contained in the fixed point set of an involution in $W(E_6)$ (a reflection in a root of E_6)) and thus $B_0 \cap B_1$ is contained in the fixed point set of an involution in S_6. Considering the involutions in S_6, we find that only the products of three commuting transpositions have a fixed point locus isomorphic to $\mathbf{P}^2$ and this $\mathbf{P}^2$ is linearly embedded. For $\sigma = (ij)(kl)(mn) \in S_6$ we have the fixed point set is

$$\mathbf{P}^2_\sigma = \{\, x_i + x_j = x_k + x_l = x_m + x_n = 0 \,\} \quad \subset S.$$

4.6. **Parametrizing** S**.** It is well-known (cf. [H], Theorem 3.1.2) that the blow up B_0 of the Segre threefold in the 10 nodes can be obtained as follows:

$$S \longleftarrow B_0 \xrightarrow{\rho_2} B_5\mathbf{P}^3 \xrightarrow{\rho_1} \mathbf{P}^3,$$

here ρ_1 is the blow up of $\mathbf{P}^3$ in 5 general points $p_1, \ldots, p_5$ and ρ_2 is the blow up of $B_5\mathbf{P}^3$ in the strict transforms of the 10 lines spanned by pairs of the points. Note that this implies that $\mathrm{rk}A^1(B_0) = 1 + 5 + 10 = 16$, in agreement with Prop. 4.3. The rational map $\mathbf{P}^3 - \to S$ is given by the linear system of quadrics which contain the 5 points.

4.7. **The Chow group of B_0 (bis).** We denote the exceptional fibers of ρ_1 by $V_1', \dots, V_5'$ and their strict transforms in B_0 by $V_1, \dots, V_5$. The divisor $V_1' \cong \mathbf{P}^2$ intersects the strict transform in $B_5\mathbf{P}^3$ of the 4 lines $\langle p_1, p_i\rangle$ $(i = 2, \dots, 5)$ in 4 points in general position. Thus each V_i is isomorphic to V.

Let $V_{ij}'' = \langle p_k, p_l, p_m\rangle$ be the plane in $\mathbf{P}^3$ spanned by the points with complementary indices, so $\{i, \dots, m\} = \{1, \dots, 5\}$. Its strict transform V_{ij}' in $B_5\mathbf{P}^3$ has divisor class $H' - V_k' - V_l' - V_m'$, here H' is the pull-back of $\mathcal{O}(1)$ on $\mathbf{P}^3$. Its strict transform V_{ij} in B_0 has class

$$V_{ij} = H - V_k - V_l - V_m - C_{kl} - C_{km} - C_{lm},$$

where C_{ij} is the exceptional divisor in B_0 over the strict transform of the line $\langle p_i, p_j\rangle$ (note that V_{ij}'' contains three of the 10 lines) and H is the pull-back of $\mathcal{O}(1)$ to B_0. The intersection point V_{ij}'' with the line $\langle p_i, p_j\rangle$ is blown up by ρ_2, so V_{ij} is also isomorphic to V.

The divisors H, $V_1, \dots V_5$, and the 10 C_{ij} are a $\mathbf{Z}$-basis of $A^1(B_0)$, so it is easy to verify that the 5 V_i's and the 10 V_{ij}'s are independent. It will be convenient to define $V_{i6} := V_i$, so we can talk simply about the 15 V_{ij}.

Using explicit formulas for the quadrics on the 5 points (cf. [H], proof of Thm. 3.2.1), one verifies that the 15 V_{ij} map to the 15 $\mathbf{P}_\sigma$. Thus we conclude, for example, that in $A^1(B_0)$ we have:

$$\hat{B}_b = \sum_{ij}^{6} V_{ij} = 10H - 5\hat{V} - 3\hat{C}_b, \qquad \text{with} \quad \hat{V} := \sum_{i=1}^{5} V_i, \quad \hat{C}_b = \sum_{kl}^{5} C_{kl}.$$

4.8. **Invariant classes in $A^1(B_0)$.** The group S_6 permutes the $\mathbf{P}_\sigma$ and thus the 15 V_{ij}'s transitively. Hence there is only a one dimensional S_6-invariant subspace in the subspace (of codimension 1) of $A^1(B_0)_{\mathbf{Q}}$ which they span. Therefore $\dim A^1(B_0)_{\mathbf{Q}}^{S_6} \leq 2$. Since also $\hat{C}_b$ is invariant and does not lie in the span of the V_{ij}'s, we have $\dim A^1(B_0)_{\mathbf{Q}}^{S_6} = 2$ and $\hat{B}_b$, $\hat{C}_b$ are a basis of the invariant classes.

4.9. **The Picard group of V.** Each of the 15 boundary surfaces V_{ij} of B_0 is isomorphic to V and it will be convenient to restrict divisor classes on B_0 (and on $\mathcal{C}$) to a V_{ij} in order to compute intersection numbers, Chern classes etc.

Since V is isomorphic to the blow up of $\mathbf{P}^2$ in 4 points, we have

$$A^1(V) \cong \mathbf{Z}l \oplus \mathbf{Z}e_1 \ldots \oplus \mathbf{Z}e_4$$

with intersection product $l^2 = 1$, $e_i^2 = -1$, $l \cdot e_i = e_i \cdot e_j = 0$ if $1 \leq i, j \leq 4$ and $i \neq j$. Moreover, with $c := e_1 + \ldots + e_4$:

$$c_1(V) = -K_V = 3l - c, \qquad c_2(V) = 7,$$

the formula for K_V is standard ([HAG], V.3.3), for c_2 one can use $\chi(\mathcal{O}_V) = 1$ and the Noether formula ([HAG], Appendix A, 4.1.2) $\chi(\mathcal{O}_V) = (K^2 + c_2)/12$.

4.10. **Lemma.** Let V_{16} $(\subset B_0)$ be the strict transform of the exceptional divisor V_1' $(\subset B_5\mathbf{P}^3)$ over $p_1 \in \mathbf{P}^3$. The restriction map on divisors is given by:

$$A^1(B_0) \longrightarrow A^1(V_{16}), \qquad \hat{B}_{b|V_{16}} = 5l - 3c, \qquad \hat{C}_{b|V_{16}} = c, \qquad V_{16}^2 = -l \qquad (\in A^1(V_{16})).$$

In particular, the restriction map is injective on S_6-invariants.

Proof. Since V_{16} maps to a point in $\mathbf{P}^3$, the class H maps to zero in $A^1(V_{16})$. Moreover, V_{16} does not intersect the divisors V_{i6} $(i = 2, \dots, 5)$ nor the 4 divisors V_{1k} with $k = 2, \dots, 5$ nor the divisors C_{kl} with $k, l \neq 1$, thus also these map to zero. The intersection V_{16} and one of the 6 divisors V_{kl}, $2 \leq k, l \leq 5$, is transversal along a divisor with class $l - e_i - e_j$ where $\{i, j, k, l\} = \{2, \dots, 5\}$. Obviously C_{1j} maps to e_{j-1}, $j = 2, \dots, 5$ (up to permutation of indices).

It remains to determine the image of $V_{16} \in A^1(B_0)$, we write V_{16}^2 for this class. Using $K_{B_0} = (-2/5)\hat{B}_b + (1/5)\hat{C}_b$, which restricts to $(-2/5)(V_{16}^2 + 6l - 2c)) + (1/5)c$, and the adjunction formula :

$$K_{V_{16}} = K_{B_0|V_{16}} + V_{16}^2 \qquad \text{we get} \quad -3l + c = \frac{3}{5}V_{16}^2 - \frac{12}{5}l + c,$$

hence $V_{16}^2 = -l$. $\square$

4.11. **Intersection numbers on B_0.** It is now easy to compute the intersection numbers. If D_1 and D_2 are S_6-invariant divisors on B_0, then $D_1 D_2 \hat{B}_b = 15 D_1 D_2 V_{16}$ since $\hat{B}_b$ is the sum of 15 surfaces in the S_6-orbit of $V := V_{16}$. Recall that we may compute the intersection product $D_1 D_2 V$ on B_0 simply as $D_{1|V} \cdot D_{2|V}$ on V, in fact:

$$D_1 \cdot D_2 \cdot V = D_1 \cdot D_2 \cdot j_* 1_V = j_*(j^*(D_1 \cdot D_2) \cdot 1_V) = j_*(j^* D_1 \cdot j^* D_2) \quad \in A^3(B_0)$$

were $j : V \hookrightarrow B_0$ is the inclusion of V in B_0 and $1_V \in A^0(V)$ is the class of V, $1_V = [V]$, and we used the projection formula and the fact that j^* is a ring homomorphism. Thus we find:

$$\hat{B}_b^3 = 15 \hat{B}_{b|V} \hat{B}_{b|V} = 15(5l - 3c)^2 = 15(25 + 9 \cdot (-4)) = 165,$$

Similarly, we get $\hat{B}_b^2 \hat{C}_b = 15(5l - 3c)c = 180$ and $\hat{B}_b \hat{C}_b^2 = 15c^2 = 60$. It remains to compute $\hat{C}_b^3$, as $\hat{C}_b$ is the sum of 10 disjoint divisors in one S_6-orbit, this is $10 C_b^3$ where C_b is a cusp divisor on B_0. But $C_b^3 = C_0^3 B_0 = 2$ by Proposition 3.2.

4.12. **Intersection numbers on $\mathcal{C}$.** Using the results on the restriction to subvarieties we can now determine some intersection numbers on the cross ratio variety as well as its canonical class.

4.13. **Theorem.** The intersection numbers between invariant classes on the cross ratio variety $\mathcal{C}$ are:

$$\hat{B}^4 = -12528 = -2^4 3^3 29, \qquad \hat{B}^3 \hat{C} = 6480 = 2^4 3^4 5, \qquad \hat{B}^2 \hat{C}^2 = -2^4 3^3 5,$$
$$\hat{B} \hat{C}^3 = 720 = 2^4 3^2 5, \qquad \hat{C}^4 = -240 = -2^4 3 \cdot 5.$$

Moreover, for boundary divisors $B_0, \dots, B_3$ which have a non-empty intersection (that is, the corresponding lines in $\mathbf{F}_3^5$ are perpendicular) we have:

$$B_0^4 = -3, \qquad B_0^3 B_1 = 1, \qquad B_0^2 B_1^2 = 1, \qquad B_0^2 B_1 B_2 = -1, \qquad B_0 B_1 B_2 B_3 = 1.$$

Proof. We follow the strategy of section 4.11, replacing the role of the surface V with that of the boundary divisor B_0. Since $\hat{B}$ is the sum of the 36 divisors in the $W(E_6)$-orbit of B_0 we get, for $W(E_6)$-invariant divisors D_1, D_2, D_3:

$$D_1 D_2 D_3 \hat{B} = 36 D_1 D_2 D_3 B_0 = 36(D_1 D_2 D_3)_{|B_0} = 36(D_{1|B_0})(D_{2|B_0})(D_{3|B_0}).$$

Using that $\hat{B}_{|B_0} = \hat{B}_b + B_0^2 = (4\hat{B}_b - 3\hat{C}_b)/5$ (cf. 4.2 and Prop. 4.3) we obtain:

$$\hat{B}^4 = 36\hat{B}^3_{|B_0} = \tfrac{36}{125}(4\hat{B}_b - 3\hat{C}_b)^3 = \tfrac{36}{125}(64\cdot(-165) - \dots) = -2^4 3^3 29,$$

similarly one finds:

$$\hat{B}^3\hat{C} = \tfrac{36}{25}(4\hat{B}_b - 3\hat{C}_b)^2\hat{C}_b = 6480, \qquad \hat{B}^2\hat{C}^2 = \tfrac{36}{5}(4\hat{B}_b - 3\hat{C}_b)\hat{C}_b^2 = -2160,$$

and $\hat{B}\hat{C}^3 = 36\hat{C}_b^3 = 720$. For $\hat{C}^4$ see Proposition 3.2.

The intersection numbers of the boundary divisors are easily determined by restriction to B_0 or $B_0B_1 = V_{ij}$ (for some i, j), the restriction map $A^1(B_0) \to A^1(V_{ij})$ is given in 4.10. Hence:

$$B_0^4 = N^3_{B_0/\mathcal{C}} = -\tfrac{1}{125}(\hat{B}_b + 3\hat{C}_b)^3 = -3, \qquad B_0^3B_1 = N^2_{B_0/\mathcal{C}}\cdot V_{ij} = \tfrac{1}{25}(5l - 3c + 3c)^2 = 1.$$

Similarly one computes the other intersection numbers. □

4.14. It is now easy to compute the degree of the moduli space of marked cubics embedded in $\mathbf{P}^9$ by the CAF-system. This computation was already done in [Fr]. Freitag used two skew $\mathbf{P}^4$'s in $\mathbf{P}^9$ (eigenspaces of a subgroup of index two of $W(E_6)$) and showed, using explicit generators of the ideal of $\mathcal{M} \subset \mathbf{P}^9$ and computer algebra, that the projection from one to the other induces a morphism $\mathcal{M} \to \mathbf{P}^4$ of degree 27.

4.15. **Corollary.** The degree of the image in $\mathbf{P}^9$ of the moduli space $\mathcal{M}$ under the CAF-map is 27.

Proof. From Theorem 2.6 it follows that the degree is given by

$$\left((\hat{B} + 3\hat{C})/4\right)^4 = \tfrac{1}{256}(\hat{B}^4 + 12\hat{B}^3\hat{C} + \dots + 81\hat{C}^4) = 27,$$

where we used the intersection numbers from Theorem 4.13. □

4.16. **The $W(E_6)$-ivariants in $A^2(\mathcal{C})$.** We already observed in 3.3 that there are 120 independent codimension 2 classes (obtained as B_iC_j) supported on the cusp divisors. It is easy to see that $W(E_6)$ permutes these classes transitively, and hence the subspace of invariant classes has dimension 1.

Next we consider the orthogonal complement (w.r.t. the intersection form) in A^2, it has dimension $147 - 120 = 27$. Classes in the complement are given for example by the selfintersections B_i^2 of the 36 boundary divisors (indeed, $B_iC_jB_k^2 = 0$, for any indices, by Proposition 3.2). A computation shows that the rank of the intersection matrix $(\,B_i^2B_j^2\,)$ is 21. The group $W(E_6)$ permutes these classes and since the permutation character is $\mathbf{1} \oplus \mathbf{15_q} \oplus \mathbf{20_p}$, we conclude that there is in $A^2(\mathcal{C})_{\mathbf{Q}}$ a 21-dimensional subspace, perpendicular to the 120 subspace, on which $W(E_6)$ acts via $\mathbf{1} \oplus \mathbf{20_p}$. In particular we found another invariant.

To get our hands on the remaining 6-dimensional subspace, we considered the image of the 306-dimensional $\mathbf{Q}$-vectorspace W generated by the $270 = 36 \cdot 15/2$ (non-empty) intersections B_iB_j of boundary divisors and the 36 B_i^2. The group $W(E_6)$ acts on W by permuting the basis vectors. Using the previous theorem (and a computer) we found that the matrix of intersection products has rank 147, hence these classes span $A^2(\mathcal{C})_{\mathbf{Q}}$ (the 270 classes B_iB_j span a subspace of codimension 1). Next one computes the kernel of $W \to A^2(\mathcal{C})_{\mathbf{Q}}$ (which is the kernel of the intersection matrix). Its perpendicular w.r.t. to the standard innerproduct on W is a $W(E_6)$-invariant subspace which maps isomorphically onto $A^2(\mathcal{C})_{\mathbf{Q}}$. Now one computes the 6-dimensional subspace which is perpendicular to the $120+21=141$-dimensional subspace we already found and one verifies that it is the standard 6-dimensional representation $\mathbf{6_p}$ of $W(E_6)$.

A class in this subspace, which corresponds to the root h, is the class $S_h = S_h^+ - S_h^-$, where S_h^+ is the sum of the $15 \cdot 4 = 60$ classes $B_{ij} \cdot B_{ijk}$ ($1 \leq i \leq j \leq 6$ and $k \neq i, j$) and S_h^- is the sum of the 60 classes $B_{ij} \cdot B_{klm}$ ($1 \leq i \leq j \leq 6$ and $\{k,l,m\} \cap \{i,j\} = \emptyset$), here we wrote B_m for the boundary divisor corresponding to the root h_m (cf. 7.4).

In any case, we can now conclude that $\mathrm{rk} A^2(\mathcal{C})^{W(E_6)} = 2$. As a byproduct, we found that the complement of the 120 classes supported on the cusps has the representation $\mathbf{1} \oplus \mathbf{20_p} \oplus \mathbf{6_p}$, which is the permutation representation of $W(E_6)$ on the 27 lines (!), cf. [F], (5.2). For completeness sake, the representation on the 120-dimensional subspace is $\mathbf{1} \oplus \mathbf{15_q} \oplus \mathbf{20_p} \oplus \mathbf{24_p} \oplus \mathbf{60_p}$.

5. The Riemann-Roch Theorem

5.1. **Riemann-Roch.** The Riemann-Roch theorem for a line bundle $\mathcal{O}_{\mathcal{C}}(D)$ on the cross ratio variety $\mathcal{C}$ is:

$$\chi(\mathcal{O}_{\mathcal{C}}(D)) = \deg\Big((1 + D + \tfrac{1}{2}D^2 + \tfrac{1}{6}D^3 + \tfrac{1}{24}D^4) \cdot (1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{24}c_1c_2 + td(\mathcal{T})_4)\Big)_4$$

where $td(\mathcal{T})_4 = \frac{-1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4)$, ([HAG], Appendix A.4) and $\mathcal{T}$ is the tangent bundle of $\mathcal{C}$. In particular, if $D = 0$ we find $1 = \chi(\mathcal{O}_{\mathcal{C}}) = td(\mathcal{T})_4$. Hence to use the Riemann-Roch theorem we only need to compute c_1 and c_2 of $\mathcal{C}$, and we will only indicate how we computed c_3 and c_4.

5.2. **Chern classes.** The Chern classes c_1 and c_2 can be computed easily by restriction to a divisor. In general, if Y is a smooth divisor on X, the exact sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{X|Y} \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

gives the following relations in $A^*(Y)$:

$$c_1(X)_{|Y} = c_1(Y) + Y^2, \quad c_2(X)_{|Y} = c_1(Y) \cdot Y^2 + c_2(Y), \quad c_3(X)_{|Y} = c_2(Y) \cdot Y^2 + c_3(Y)$$

where $Y^2 = c_1(N_{Y/X}) \in A^1(Y)$, the first relation is equivalent to the adjunction formula.

5.3. **Proposition.** The Chern classes of a boundary divisor B_0 are:

$$c_1(B_0) = (2\hat{B}_b + \hat{C}_b)/5, \qquad c_2(B_0) = (4\hat{B}_b^2 - 36\hat{C}_b^2)/25, \qquad c_3(B_0) = 34.$$

Proof. To compute $c_2(B_0)$ we consider the divisor $Q \cong (\mathbf{P}^1)^2$ which lies over a cusp in the Segre threefold. We can identify Q with the divisor $D_1 = B_0 \cap C_0$ as in Proposition 3.2. Only six of the nine boundary divisors, besides B_0, which meet C_0 intersect Q, they intersect it in $3D_1D_2 + 3D_1D_3 = 3\Delta$ where Δ is the diagonal of $(\mathbf{P}^1)^2$. Thus $\hat{B}_b$ restricts to 3Δ on Q. Since $\hat{C}_b$ is the sum of the 10 disjoint cusp divisors on B_0, one of which is Q, its restriction to Q is $Q^2 = N_{Q/B_0}$. Since $K_Q = -2\Delta$ the adjunction formula gives

$$-2\Delta = K_{B_0|Q} + Q^2 = \Big((-2\hat{B}_b - \hat{C}_b)/5\Big)_{|Q} + Q^2 = (-6\Delta + 4Q^2)/5$$

hence $Q^2 = -\Delta$. Furthermore, since $\hat{B}_b^2\hat{C}_b = 10 \cdot 18$, $\hat{C}_b^3 = 20$ and $\hat{C}_b$ is the sum of 10 disjoint copies of Q, we get $\hat{B}_b^2 Q = 18$, $\hat{C}_b^2 Q = 2$.

Now we write $c_2(B_0) = n\hat{B}_b^2 + m\hat{C}_b^2$ and use the relation 5.2 for $Q \subset B_0$:

$$(n\hat{B}_b^2 + m\hat{C}_b^2)_{|Q} = c_1(Q)Q^2 + c_2(Q) = (2\Delta)(-\Delta) + 4 = 0,$$

hence $18n + 2m = 0$. On any smooth threefold we have $c_1c_2/24 = \chi(\mathcal{O})$. Since $\chi(\mathcal{O}) = 1$ for B_0 this implies

$$24 = \tfrac{1}{5}(2\hat{B}_b + \hat{C}_b)(n\hat{B}_b^2 + m\hat{C}_b^2) = -30n - 20m.$$

From these equations one finds $n = 4/25$ and $m = -36/25$.

Recall that B_0 is obtained as a blow up in two different ways. One is described in section 4.6, the other in the proof of Proposition 4.3. In [Fu1], Thm. 15.4 the Chern classes of the blow up of a smooth threefold Y in a point p or in a smooth curve X are given. Applying these formulas allows one to compute, in two different ways, $c_3(B_0)$ and we found the same result, $c_3(B_0) = 34$, in both ways. This method also allowed us to verify the formula for $c_2(B_0)$ determined above. □

5.4. **Theorem.** The Chern classes of the cross ratio variety $\mathcal{C}$ are:

$$c_1(\mathcal{C}) = (\hat{B} - \hat{C})/4, \qquad c_2(\mathcal{C}) = (\hat{B}^2 - 9\hat{C}^2)/8, \qquad c_3(\mathcal{C}) = \tfrac{13}{3\cdot 96}\hat{B}^3 + \tfrac{181}{96}\hat{C}^3,$$

and $c_4(\mathcal{C}) = 271$.

Proof. Since $c_1(\mathcal{C}) = -K_{\mathcal{C}}$, the first result follows from Theorem 2.4. The expression for c_2 follows from the formula for $c_2(X)_{|Y}$ in 5.2 applied to $B_0 \subset \mathcal{C}$ and the previous proposition. Similarly, since the Chern class c_3 is invariant under $W(E_6)$ it can be written as $a\hat{B}^3 + b\hat{C}^3$ for certain a, $b \in \mathbf{Q}$. Restricting to B_0 and to a cusp divisor C_0 and using $c_3(B_0) = 34$, $c_3(C_0) = 8$ (which follows from $C_0 \cong (\mathbf{P}^1)^3$) we find two equations for a and b from which c_3 can be determined. The class c_4 can be computed from the other c_i and the relation $td(\mathcal{T})_4 = 1$. □

6. Applications

6.1. **The CAF-linear system.** In [Fr], Freitag determined the Hilbert function of the graded ring $\mathbf{C}[Y_0, \dots, Y_9]/\mathcal{I}$, where $\mathcal{I}$ is the homogeneous ideal defining $\mathcal{M} \subset \mathbf{P}^9$. The Hilbert function coincides with the map $n \mapsto \chi(\mathcal{O}_{\mathcal{C}}(nH))$, see the theorem below. This suggests that that the multiplication maps $Sym^n H^0(H) \to H^0(nH)$ are surjective and that $H^i(nH) = 0$ for $i, n > 0$.

6.2. **Theorem.** Let $H = (\hat{B} + 3\hat{C})/4$ be the pull-back of the hyperplane section of $\mathcal{C} \to \mathbf{P}^9$. Then we have:

$$\chi(\mathcal{O}_{\mathcal{C}}(nH)) = \tfrac{9}{8}n^4 + \tfrac{9}{4}n^3 + \tfrac{27}{8}n^2 + \tfrac{9}{4}n + 1.$$

Proof. This follows by direct computation from the Riemann-Roch formula. □

6.3. **Tritangent divisors.** We haven't used the tritangent divisors to obtain the results above. However, applying the same methods we can describe the Chow groups of a tritangent divisor. We will denote the tritangent divisor in $\mathcal{C}$ obtained from $\mathbf{P}^3_{\mathrm{w}}$ in $\tilde{T}'$ by T_0. The strict transform of the hyperplane class in $\mathbf{P}^3_{\mathrm{w}}$ will be denoted by $H_{\mathrm{w}} \in A^1(T_0)$. There is an identification $\mathbf{P}^3_{\mathrm{w}} \cong \mathbf{P}(Q(F_4) \otimes_{\mathbf{Z}} \mathbf{C})$, where $Q(F_4)$ is the root lattice of the root system system of type F_4 such that the action of $W(F_4)$ on $Q(F_4)$ induces the biregular action of $W(F_4)$ $(\subset W(E_6))$ on T_0 (cf. [vG], 6.4, 6.5). The root system F_4 has 24 short roots, these are the elements of S, and

24 long roots, the elements of R, these form a subroot system of type D_4 in F_4. The 12 points in $\mathbf{P}(Q(F_4)\otimes_{\mathbf{Z}}\mathbf{C})$ defined by the short roots of F_4 are blown up to divisors in T_0. The sum of these 12 divisors in T_0 will be denoted by $\hat{B}_{t,e}$.

6.4. **Proposition.** The Chow groups of a tritangent divisor T_0 are:

$$A^1(T_0)\cong A^2(T_0)\cong \mathbf{Z}^{29}.$$

Moreover we have:

$$A^1(T_0)^{W(F_4)}\cong A^2(T_0)^{W(F_4)}\cong \mathbf{Z}^3$$

and the classes H_{w}, $\hat{B}_t:=\hat{B}_{|T_0}$ and $\hat{C}_t:=\hat{C}_{|T_0}$ are a basis of $A^1(T_0)^{W(F_4)}_{\mathbf{Q}}$.

The chern classes of T_0 are:

$$-c_1(T_0)=K_{T_0}=2H_{\mathrm{w}}-\frac{1}{2}\hat{B}_t-\frac{1}{2}\hat{C}_t,\qquad c_2(T_0)=6H_{\mathrm{w}}^2-2H_{\mathrm{w}}\hat{C}_t-\hat{C}_t^2,\qquad c_3(T_0)=92.$$

The hyperplane class of the CAF-map restricts to

$$3H_{\mathrm{w}}-\hat{B}_{t,e}.$$

Proof. The 12 curves $C_i'\subset\tilde{T}'$ which are blown up in π' intersect the projective space $\mathbf{P}^3_{\mathrm{w}}$ in 12 points, these are the points defined by S. This adds 12 to the rank of A^1. Next in $\tilde{T}''$ the strict transform of $\mathbf{P}^3_{\mathrm{w}}$ meets the 16 S_i'''s in 16 smooth rational curves (these curves are the strict transforms of 16 lines in $\mathbf{P}^3_{\mathrm{w}}$, each line contains three of the points, on each point there are 4 lines). Thus for the strict transform $\hat{E}$ of $\mathbf{P}^3_{\mathrm{w}}$ in $\hat{T}$ we get $\mathrm{rk}A^1(\hat{E})=1+12+16=29$.

The intersection of $\hat{E}$ and a $\hat{C}_i\cong\mathbf{P}^1\times V$ is a copy of V (cf. [N], p. 22). Since the contraction $r:\hat{T}\to\mathcal{C}$ induces the projection $\mathbf{P}^1\times V\to V$, the map r induces an isomorphism $\hat{E}\cong T_0$. Thus also $\mathrm{rk}A^1(T_0)=29$. The rank of A^2 follows since T_0 is easily seen to be smooth by construction. The invariant classes are also obvious.

A tritangent divisor has non-empty intersection with each boundary divisor. The 12 divisors in $\tilde{T}'$ obtained from the subtori $\alpha=1$ in T, with α a root of D_4, intersect $\mathbf{P}^3_{\mathrm{w}}$ in linear subspaces defined by the same root of F_4 ([vG], proof of Thm. 6.5). We denote by $\hat{B}_{t,i}$ the sum of the twelve divisors in T_0 obtained in this way. The other $36-12=2\cdot 12$ boundary divisors in $\mathcal{C}$ are obtained from the $V(\tau)$ where $\tau\in S$, these are paired by $\tau\leftrightarrow-\tau$. The image C_j in $\tilde{T}$ of each of the twelve curves C_j' in $\tilde{T}'$ has non-empty intersection with exactly two such boundary divisors, the two are in a pair and each intersection is transversal in one point. We denote the pair defined by C_j by $\pm\tau_j$. The divisors $\hat{T}$ and $\widehat{V(\pm\tau_j)}$ are still disjoint, but each meets the divisor $\hat{C}_j\cong C_j\times V$ in a copy of V. In the final blow down $r:\hat{T}\to\mathcal{C}$, $\hat{C}_j$ is contracted onto V, hence this surface is the intersection of any two of T_0, B_{τ_j}, and $B_{-\tau_j}$ where B_τ is the boundary divisor defined by $V(\tau)$. We denote the sum of these 12 surfaces by $\hat{B}_{t,e}$, then

$$\hat{B}_t=\hat{B}_{t,i}+2\hat{B}_{t,e}.$$

Using the divisor of $\lambda-1$ (cf. 7.1) restricted to T_0 we find that T_0^2 is is the sum of (-1) times one of the twelve divisors in $\hat{B}_{t,i}$ (it is $D^1_\lambda\cap T_0)$), minus the sum of 4 divisors in $\hat{C}_t$ and plus the sum of 6 divisors from $\hat{B}_{t,e}$. Symmetrizing w.r.t. $W(F_4)$ (so we replace each divisor in $\hat{B}_{t,i}$ by $(1/12)\hat{B}_{t,i}$ etc.) we find:

$$T_0^2 = \frac{-1}{12}\hat{B}_{t,i} + \frac{6}{12}\hat{B}_{t,e} - \frac{4}{16}\hat{C}_t = \frac{-1}{12}\hat{B}_{t,i} + \frac{1}{2}\hat{B}_{t,e} - \frac{1}{4}\hat{C}_t.$$

The canonical divisor of T_0 can be obtained from the adjunction formula for $T_0 \subset \mathcal{C}$:

$$K_{T_0} = K_{\mathcal{C}|T_0} + T_0^2 = \frac{1}{4}(-\hat{B}_{t,i} - 2\hat{B}_{t,e} + \hat{C}_t) - \frac{1}{12}\hat{B}_{t,i} + \frac{1}{2}\hat{B}_{t,e} - \frac{1}{4}\hat{C}_t = \frac{-1}{3}\hat{B}_{t,i}.$$

The canonical divisor of T_0 can also be computed from the construction of T_0 given above:

$$K_{T_0} = -4H_{\mathrm{w}} + 2\hat{B}_{t,e} + \hat{C}_t.$$

Thus we find:

$$\hat{B}_{t,i} = 12H_{\mathrm{w}} - 6\hat{B}_{t,e} - 3\hat{C}_t.$$

The $12H_{\mathrm{w}}$ in this formula arises from the fact that each of the twelve divisors in $\hat{B}_{t,i}$ is the strict transform of plane in $\mathbf{P}^3_{\mathrm{w}}$. Since $\hat{B}_t = \hat{B}_{t,i} + 2\hat{B}_{t,e}$, this relation implies $4\hat{B}_{t,e} = 12H_{\mathrm{w}} - \hat{B}_t - 3\hat{C}_t$ and thus we can eliminate $\hat{B}_{t,e}$ from the last formula for K_{T_0} to obtain the formula in the theorem. The other Chern classes can be computed by restricting the Chern classes of $\mathcal{C}$ or directly by using the blow up formulas as in the proof of 5.3.

The restriction of the CAF-hyperplane class to T_0 is:

$$\tfrac{1}{4}(\hat{B} + 3\hat{C})_{|T_0} = \frac{1}{4}(\hat{B}_{t,i} + 2\hat{B}_{t,e} + 3\hat{C}_t) = 3H_{\mathrm{w}} - \hat{B}_{t,e}$$

which agrees with the fact that the CAF-map, when restricted to $\mathbf{P}^3_{\mathrm{w}}$, is given by the cubics which vanish on the 12 points which are blown up to give $\hat{B}_{t,e}$ ([vG], Thm 6.5). $\square$

6.5. **Remark.** The tritangent divisor T_0 also has non-empty intersection with the other $45-1 = 12 + 2\cdot 16$ tritangent divisors. It is verified in [vG], 6.7 that $2\cdot 16$ tritangent divisors intersect T_0 in the strict transform of 16 quadrics in $\mathbf{P}^3_{\mathrm{w}}$.

The other 12 tritangent divisors intersect $\mathbf{P}^3_{\mathrm{w}}$ in the 12 linear subspaces defined by the short roots, that is by a pair $\pm\tau \in S$. The involutions corresponding to these 12 divisors are just the reflections in the corresponding short roots in $W(F_4) \subset W(E_6)$ ([vG], proof of Thm 6.5). The involution defined by τ permutes the boundary divisors $B_{\pm\tau}$ (defined by $V(\pm\tau) \subset \tilde{T}$) in $\mathcal{C}$. Thus we conclude that the surface $V = T_0 \cap B_\tau \cap B_{-\tau}$ must coincide with the divisor in T_0 defined by the plane in $\mathbf{P}^3_{\mathrm{w}}$ defined by the short root τ. In particular, V lies in two tritangent as well as in two boundary divisors.

The fact that there are surfaces in the intersection of pairs of boundary and tritangent divisors can also easily be seen using the ball quotient description of $\mathcal{C}$. Following [AF], the ball $\mathcal{B}$ is viewed as a subvariety of $\mathbf{C}^5$:

$$\mathcal{B} = \{(1, z_1, z_2, z_3, z_4) \in \mathbf{C}^5 : \ 1 - |z_1|^2 - \ldots - |z_4|^2 > 0\,\}$$

and we consider the hermitian form $(z, w) = z_0\bar{w}_0 - z_1\bar{w}_1 - \ldots - z_4\bar{w}_4$ on $\mathbf{C}^5$. Let $e_1 = (0,1,0,0,0)$, $e_2 = (0,0,1,0,0)$, then the subvarieties of the ball defined by $(z, e_i) = 0$ map onto tritangent divisors T_i under the map

$$\mathcal{B} \longrightarrow \mathcal{B}/\Gamma \cong \mathcal{C} - \{\text{cusp divisors}\}.$$

The subvarieties defined by $w_1 = e_1 + e_2$ and $w_2 = e_1 - e_2$ map onto boundary divisors T_j. Obviously, the intersection of these four divisors is a surface.

7. Appendix

7.1. **The divisor of λ.** The divisor of the rational function λ, as well as $\lambda - 1$, are used to obtain relations in the various Chow groups considered in this paper. In the toric variety $\tilde{T}$ we have

$$(\lambda) = \sum_{\tau \in S \cup R} n_\tau V(\tau) \qquad \text{with} \quad n_\tau = (\epsilon_1 - \epsilon_2, \tau)$$

(standard inner product on $\mathbf{R}^4$). Obviously, we have $n_{-\tau} = -n_\tau$, so it suffices to list the $\tau \in R \cap S$ with $n_\tau > 0$ which is easily done.

$$n_\tau > 0 \quad \text{iff} \quad S_{\lambda,0} \cup R_{\lambda,0} \cup \{\epsilon_1 - \epsilon_2\},$$

where

$S_{\lambda,0} = \{\epsilon_1, -\epsilon_2, (\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)/2, (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2, (\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/2, (\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)/2\}$,

here $\tau \in S$ so these give boundary divisors,

$$R_{\lambda,0} = \{\epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_1 - \epsilon_4, -\epsilon_2 + \epsilon_3, -\epsilon_2 - \epsilon_3, -\epsilon_2 + \epsilon_4, -\epsilon_2 - \epsilon_4\}$$

here $\tau \in R$ so these give cusp divisors, and finally $n_\tau = 2$ if $\tau = \epsilon_1 - \epsilon_2$, the divisor $V(\epsilon_1 - \epsilon_2)$ is also a cusp divisor.

To find the divisor of the rational function λ on $\mathcal{C}$, we observe that a divisor $V(\tau)$ does not intersect T, hence it does not contain $e \in T$ (which gets blown up in $\tilde{T}'$) nor any of the 12 curves in $\tilde{T}'$ which are blown up in $\tilde{T}''$ nor any of the 16 surfaces in $\tilde{T}''$ which are blown up in $\tilde{T}$. Thus the divisor of λ in $\hat{T}$ is simply the sum of the strict transforms of the divisors $V(\tau) \subset \tilde{T}$ with the same multiplicities. Finally, the direct image under r of this divisor in $\mathcal{C}$ is the divisor of λ on $\mathcal{C}$.

Using the tables from 7.4 below one finds that

$S_{\lambda,0} = \{h_{345}, h_{13}, h_{136}, h_{26}, h_{246}, h_{256}\} \subset \{h_{345}, h_{13}, h_{145}, h_{136}, h_{26}, h_{123}, h_{246}, h_{256}, h_{45}\} = [13.45.26]$,

that is, this set of 6 positive roots can be completed to a set three orthogonal A_2's (see 7.6). Similarly one finds

$S_{\lambda,\infty} = \{h_{12}, h_{245}, h_{36}, h_{126}, h_{356}, h_{346}\} \subset \{h_{12}, h_{245}, h_{145}, h_{36}, h_{126}, h_{123}, h_{356}, h_{346}, h_{45}\} = [12.45.36]$,

so, using the *same* three roots h_{145}, h_{123}, h_{45}, also this set can be completed to a set of three orthogonal A_2's. These sets of nine roots correspond to the cusp divisors D_v where $v = (1, -1, 1, 0, 0) \in N_0$ (with $D_v = V(\epsilon_1 - \epsilon_2)$) and $w = (1, 1, -1, 0, 0)$ (with $D_w = V(-\epsilon_1 + \epsilon_2)$) respectively. For $z \in N_0$ we define $\{z\}$ to be the set of nine (positive) roots in the corresponding A_2^3 and we put

$$B_z = \sum_{t \in \{z\}} D_t.$$

Next we consider the cusp divisors. Using the table 7.6 we find that the eight $\tau \in R_{\lambda,0}$ correspond to the lines generated by the eight vectors $(1, -1, 0, \pm 1, 0)$, $(1, -1, 0, 0, \pm 1)$, $(1, 0, 1, \pm 1, 0)$, $(1, 0, 1, 0, \pm 1)$. These vectors are all perpendicular to w, similarly the $\tau \in R_{\lambda,\infty}$ correspond to the vectors $(1, 1, 0, \pm 1, 0)$, $(1, 1, 0, 0, \pm 1)$, $(1, 0, -1, \pm 1, 0)$, $(1, 0, -1, 0, \pm 1)$ which are all in $v^\perp$. The linear space $v^\perp \cong \mathbf{F}_3^4$ contains 13 isotropic lines, 8 of these are spanned by the vectors listed, another 4 are spanned by $(0,1,1,\pm1,0)$ and $(0,1,1,0,\pm1)$ (note that these vectors are also in $w^\perp$) and finally there is the line spanned by v itself. We write

$$C_{v^\perp} = \sum_{z\in v^\perp} D_z$$

for the sum of these 13 cusp divisors. Putting all this together, we have $(\lambda) = B_v + C_{w^\perp} + 3D_v - (B_w + C_{v^\perp} + 3D_w)$.

As a byproduct of our labors we get:

$$E_v = E_w, \quad (\in A^1(\mathcal{C})) \qquad \text{with} \quad E_v = B_v - C_{v^\perp} + 3D_v.$$

The group $W(E_6)$ permutes the $v \in N_0$ and it easily follows that any two E_v's are linearly equivalent, hence E_v is an invariant class. Symmetrizing E_v gives:

$$E_v = \frac{9}{36}\hat{B} + \frac{(-13+3)}{40}\hat{C} = (\hat{B} - \hat{C})/4 = -K_{\mathcal{C}}.$$

7.2. **The divisor of $\lambda - 1$.** The rational function $\lambda - 1$ on has the same poles as λ. Its divisor of zeroes $(\lambda - 1)$ in $\tilde{T}$ is the closure in $\tilde{T}$ of the subtorus defined by $\lambda = 1$ in T, we denote it by D^1_λ. Since $\lambda(e) = 1$, the divisor $(\lambda - 1)_0$ in $\tilde{T}'$, the blow up $\tilde{T}$ of e, is given by the sum of the strict transform of D^1_λ and the exceptional fibre $\mathbf{P}^3_{\mathrm{w}}$. The strict transforms in $\hat{T}$ of the exceptional divisors for the map $\pi' : \tilde{T}'' \to \tilde{T}'$ are blown down by $r : \hat{T} \to \mathcal{C}$, hence in the end they do not contribute to $(\lambda - 1)_0$ in $A^1(\mathcal{C})$. The map π'' blows up 16 surfaces in $\tilde{T}'$, these are the strict transforms of surfaces in $\tilde{T}$. By inspection one finds that 4 of these are contained in D^1_λ, to be explicit, the defining equations of these curves are:

$$\lambda = \mu\nu\rho = 1, \qquad \lambda = \mu\rho = 1, \qquad \lambda = \nu\rho = 1, \qquad \lambda = \rho = 1.$$

Therefore the divisor $(\lambda - 1)_0$ in $\hat{T}$ has 6 components (with multiplicity one), the strict transforms of D^1_λ $(\subset \tilde{T})$, of $\mathbf{P}^3_{\mathrm{w}})$ $(\subset \tilde{T}')$ and the exceptional divisors over the 4 surfaces. The divisor $(\lambda - 1)_0$ in $\mathcal{C}$ is the direct image of this divisor, in particular it is the sum of one boundary divisor (coming from D^1_λ), one tritangent divisor (coming from $\mathbf{P}^3_{\mathrm{w}}$) and 4 cusp divisors (coming from the exceptional divisors over the surfaces).

7.3. **Tables.** We collect some of the notations used by various authors for referring to tritangents, boundary and cusp divisors (cf. [Fr2] and [S]). We also give a $W(E_6)$-equivariant map between the sets of these divisors and the sets N_i (see 1.7).

7.4. **The boundary divisors.** The 36 boundary divisors are parametrized by the 36 positive roots of E_6. These roots generate the root lattice $Q(E_6) \cong \mathbf{Z}^6$. The quadratic form given by the Cartan matrix on $Q(E_6)$ has determinant 3, hence the $\mathbf{F}_3$-valued bilinear form it induces on FF_3^6 is degenerate. In fact, $v = \alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 = h_{12} - h_{23} + h_{45} - h_{56} \in Q(R)$ has the property that $(v, \alpha) \in 3\mathbf{Z}$ (this is easy to verify on a basis of simple roots of E_6 using the Dynkin diagram), but $v \notin 3Q(R)$. More intrinsically, we note that $v/3$ lies in the weight lattice $P(E_6)$ and that $P(E_6) = Q(E_6) + \mathbf{Z}(v/3)$, hence $Q(E_6)/(3Q(E_6) + \mathbf{Z}v) \cong Q(E_6)/3P(E_6)$ (cf. [B], Exercises, VI §4, n. 2). On $\mathbf{F}_3^5 \cong Q(E_6)/(3P(E_6)$ we get a non-degenerate $\mathbf{F}_3$-valued quadratic form q and we have an induced homomorphism $W(E_6) \to O(\mathbf{F}_3^5, q)$.

Under the quotient map

$$\pi : Q(R) \longrightarrow Q(E_6)/(3Q(E_6) + \mathbf{Z}v) \cong \mathbf{F}_3^5$$

the roots of E_6 map to elements of N_{-1}. We fix π by defining the images of the basis of simple roots h_{12}, h_{123}, h_{23}, h_{34}, h_{45}, h_{56}:

$$\pi(h_{i,i+1}) = f_i - f_{i+1} \qquad (1 \le i \le 4), \qquad \pi(h_{123}) = (0,0,0,1,1),$$

here f_i is the i-th standard basis vector of $\mathbf{F}_3^5$, and in view of $\pi(v) = 0$ we get

$$\pi(h_{56}) = h_{12} - h_{23} + h_{45} = (1,1,1,1,-1).$$

The first table lists the roots of $D_4 \subset E_6$, the corresponding characters of T, the name of the root in E_6 and its image in $\mathbf{P}(\mathbf{F}_3^5)$. The last three tables list the elements in S, which define boundary divisors in $\mathcal{C}$, the corresponding positive root in E_6 and its image in $\mathbf{P}(\mathbf{F}_3^5)$. Note that $(\pm\pm\pm\pm)$ stands for the vector $(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2$.

$e_3 + e_4$	μ	h_{123}	$(0,0,0,1,1)$
$e_3 - e_4$	ν	h_{45}	$(0,0,0,1,-1)$
$e_2 + e_4$	$\mu\rho$	h_{124}	$(0,0,1,0,1)$
$e_2 - e_4$	$\nu\rho$	h_{35}	$(0,0,1,0,-1)$
$e_2 + e_3$	$\mu\nu\rho$	h_{125}	$(0,0,1,1,0)$
$e_2 - e_3$	ρ	h_{34}	$(0,0,1,-1,0)$
$e_1 + e_4$	$\lambda\mu\rho$	h_{134}	$(0,1,0,0,1)$
$e_1 - e_4$	$\lambda\nu\rho$	h_{25}	$(0,1,0,0,-1)$
$e_1 + e_3$	$\lambda\mu\nu\rho$	h_{135}	$(0,1,0,1,0)$
$e_1 - e_3$	$\lambda\rho$	h_{24}	$(0,1,0,-1,0)$
$e_1 + e_2$	$\lambda\mu\nu\rho^2$	h_{145}	$(0,1,1,0,0)$
$e_1 - e_2$	λ	h_{23}	$(0,1,-1,0,0)$

ϵ_4	h_{234}	$(1,0,0,0,1)$
$-\epsilon_4$	h_{15}	$(1,0,0,0,-1)$
ϵ_3	h_{235}	$(1,0,0,1,0)$
$-\epsilon_3$	h_{14}	$(1,0,0,-1,0)$
ϵ_2	h_{245}	$(1,0,1,0,0)$
$-\epsilon_2$	h_{13}	$(1,0,-1,0,0)$
ϵ_1	h_{345}	$(1,1,0,0,0)$
$-\epsilon_1$	h_{12}	$(1,-1,0,0,0)$

$(----)$	h	$(1,1,1,1,1)$
$(---+)$	h_{56}	$(1,1,1,1,-1)$
$(--+-)$	h_{46}	$(1,1,1,-1,1)$
$(--++)$	h_{456}	$(1,1,1,-1,-1)$
$(-+--)$	h_{36}	$(1,1,-1,1,1)$
$(-+-+)$	h_{356}	$(1,1,-1,1,-1)$
$(-++-)$	h_{346}	$(1,1,-1,-1,1)$
$(-+++)$	h_{126}	$(1,1,-1,-1,-1)$

$(+---)$	h_{26}	$(1,-1,1,1,1)$
$(+--+)$	h_{256}	$(1,-1,1,1,-1)$
$(+-+-)$	h_{246}	$(1,-1,1,-1,1)$
$(+-++)$	h_{136}	$(1,-1,1,-1,-1)$
$(++--)$	h_{236}	$(1,-1,-1,1,1)$
$(++-+)$	h_{146}	$(1,-1,-1,1,-1)$
$(+++-)$	h_{156}	$(1,-1,-1,-1,1)$
$(++++)$	h_{16}	$(1,-1,-1,-1,-1)$

7.5. **Tritangents.** The tritangents were labelled by Cayley, later a more comprehensible notation was introduced by Schläfli. The 27 lines on a smooth cubic surface are denoted by $a_1, \dots, a_6$ (these are 6 skew lines which can thus be blown down to points p_i), $b_1, \dots, b_6$ (b_i maps to the conic on the 6 points except for p_i) and c_{ij}, $1 \le i < j \le 6$ (c_{ij} maps to the line $\langle p_i, p_j \rangle$). The 45 tritangent planes are denoted by

$$(ij) = \langle a_i, b_j, c_{ij} \rangle, \qquad (ij.kl.mn) = \langle c_{ij}, c_{kl}, c_{mn} \rangle,$$

here $i \ne j$ and $(ij) \ne (ji)$, also $\{i, \dots, n\} = \{1, \dots, 6\}$ and $i < j$, $k < l$, $m < n$. A correspondence between Cayley's labels and those of Schläfli was given in [S] and is copied here.

Each tritangent plane is determined by a $D_4 \subset E_6$ (cf. [vG], 1.8). The span of the image of such a D_4 in $\mathbf{F}_3^5$ has codimension one (the image of the 'standard' D_4 spanned by

h_{123}, h_{23}, h_{34}, h_{45} has this property, the other D_4's are in $W(E_6)$-orbit of this one), hence it is the perpendicular of a one dimensional subspace. In this way each tritangent defines a point in $\mathbf{P}(\mathbf{F}_3^5)$, one checks that this point lies in N_1 for the standard D_4, and thus it lies in N_1 for all tritangents. Alternatively, one can use [AF] to find this correspondence.

Another way to describe this correspondence was found by Matsumoto and Terasoma [MT], Prop. 3.4. They identify $\mathbf{F}_3^5$ with a subgroup of the 3-torsion points of the intermediate Jacobian of a cubic threefold associated to a cubic surface.

ξ	(56)	$(0,0,0,0,1)$
x	(46)	$(0,0,0,1,0)$
x	(36)	$(0,0,1,0,0)$
$\overline{\mathrm{x}}$	(26)	$(0,1,0,0,0)$
w	(16)	$(1,0,0,0,0)$
$\overline{\mathrm{y}}$	(61)	$(0,1,1,1,1)$
$\overline{\mathrm{z}}$	(15)	$(0,1,1,1,-1)$
z	(14)	$(0,1,1,-1,1)$
y	(16.23.45)	$(0,1,1,-1,-1)$
z	(13)	$(0,1,-1,1,1)$
y	(16.24.35)	$(0,1,-1,1,-1)$
η	(16.25.34)	$(0,1,-1,-1,1)$
ζ	(12)	$(0,1,-1,-1,-1)$
r_1	(62)	$(1,0,1,1,1)$
$\overline{\mathrm{q}}_1$	(25)	$(1,0,1,1,-1)$
$\overline{\mathrm{m}}_1$	(24)	$(1,0,1,-1,1)$
n_1	(13.26.45)	$(1,0,1,-1,-1)$
$\overline{\mathrm{n}}$	(23)	$(1,0,-1,1,1)$
m	(14.26.35)	$(1,0,-1,1,-1)$
$\overline{\mathrm{q}}$	(15.26.34)	$(1,0,-1,-1,1)$
$\overline{\mathrm{r}}$	(21)	$(1,0,-1,-1,-1)$

$\overline{\mathrm{n}}_1$	(63)	$(1,1,0,1,1)$
m_1	(35)	$(1,1,0,1,-1)$
q_1	(34)	$(1,1,0,-1,1)$
r_1	(12.36.45)	$(1,1,0,-1,-1)$
r	(32)	$(1,-1,0,1,1)$
q	(14.25.36)	$(1,-1,0,1,-1)$
$\overline{\mathrm{m}}$	(15.24.36)	$(1,-1,0,-1,1)$
n	(31)	$(1,-1,0,-1,-1)$
$\overline{\mathrm{l}}$	(64)	$(1,1,1,0,1)$
l	(45)	$(1,1,1,0,-1)$
$\overline{\mathrm{g}}$	(43)	$(1,1,-1,0,1)$
$\overline{\mathrm{h}}$	(12.35.46)	$(1,1,-1,0,-1)$
h	(42)	$(1,-1,1,0,1)$
g	(13.25.46)	$(1,-1,1,0,-1)$
$\overline{\mathrm{l}}_1$	(15.23.46)	$(1,-1,-1,0,1)$
l_1	(41)	$(1,-1,-1,0,-1)$
$\overline{\mathrm{p}}$	(65)	$(1,1,1,1,0)$
p	(54)	$(1,1,1,-1,0)$
f	(53)	$(1,1,-1,1,0)$
θ	(12.34.56)	$(1,1,-1,-1,0)$
$\overline{\theta}$	(52)	$(1,-1,1,1,0)$
$\overline{\mathrm{f}}$	(13.24.56)	$(1,-1,1,-1,0)$
p_1	(14.23.56)	$(1,-1,-1,1,0)$
$\overline{\mathrm{p}}_1$	(51)	$(1,-1,-1,-1,0)$

7.6. **The cusp divisors.** The 40 cusp divisors can be labelled by 'triads of trihedral pairs' (cf. [H], 6.1.1), they correspond also to copies of three orthogonal A_2's in E_6 (cf. [H], 6.1.5.3) The notation we use is:

$$[ijk.lmn] = \begin{bmatrix} h_{ij} & h_{jk} & h_{ik} \\ h_{lm} & h_{mn} & h_{ln} \\ h & h_{ijk} & h_{lmn} \end{bmatrix}, \qquad [ij.kl.mn] = \begin{bmatrix} h_{ij} & h_{ikl} & h_{jkl} \\ h_{kl} & h_{kmn} & h_{lmn} \\ h_{mn} & h_{nij} & h_{mij} \end{bmatrix},$$

here the rows of the matrices are thee positive vectors of E_6 which span an A_2 and the roots in different rows are perpendicular.

The image under π of the span of an $A_2 \subset Q(E_6)$ is a two dimensional subspace of $\mathbf{F}_3^5$, its projectivizations thus has 4 points, three of which are lines spanned by the positive roots, the other line is spanned by an element from N_0. The span of the images of three perpendicular

A_2's has codimension one in $\mathbf{F}_3^5$ (in fact, the span of each of the three contains the same element v from N_0). Hence this subspace is the perpendicular of an element, which in fact is this $v \in N_0$ (to check this it suffices to verify this statement for one of the 40 triples, and to apply $W(E_6)$ to obtain the result in general). Conversely, given a vector $v \in N_0$, it is perpendicular to exactly 9 elements in N_{-1}.

The first table lists the equations of a subtorus of T whose closure in the toric variety $\tilde{T}$ is one of the 16 surfaces S_i, the pair of roots in $D_4 \subset E_6$ which define the surface are also given. This pair of roots spans an $A_2 \subset E_6$ which is contained in a unique triple of orthogonal A_2's. The corresponding cusp divisor in $\mathcal{C}$ is then labelled as above as well as by a vector in N_0, this vector is normalized by its first non-zero component being $+1$. The last table lists a $\tau \in R$, which defines a cusp divisor in $\mathcal{C}$, a label for the corresponding triple of orthogonal A_2's as well as the corresponding normalized vector in N_0.

$\nu = \rho = 1$	$h_{34},\, h_{45}$	[126.345]	$(0,0,1,1,1)$
$\mu = \rho = 1$	$h_{34},\, h_{123}$	[34.12.56]	$(0,0,1,1,-1)$
$\mu = \nu\rho = 1$	$h_{35},\, h_{123}$	[35.12.46]	$(0,0,1,-1,1)$
$\nu = \mu\rho = 1$	$h_{45},\, h_{124}$	[45.12.36]	$(0,0,-1,1,1)$
$\nu = \lambda\rho = 1$	$h_{24},\, h_{45}$	[136.245]	$(0,1,0,1,1)$
$\mu = \lambda\rho = 1$	$h_{24},\, h_{123}$	[24.13.56]	$(0,1,0,1,-1)$
$\mu = \lambda\nu\rho = 1$	$h_{123},\, h_{25}$	[25.13.46]	$(0,1,0,-1,1)$
$\nu = \lambda\mu\rho = 1$	$h_{45},\, h_{134}$	[45.13.26]	$(0,1,0,-1,-1)$
$\lambda = \nu\rho = 1$	$h_{23},\, h_{35}$	[146.235]	$(0,1,1,0,1)$
$\lambda = \mu\rho = 1$	$h_{23},\, h_{124}$	[23.14.56]	$(0,1,1,0,-1)$
$\lambda\nu\rho = \mu\rho = 1$	$h_{25},\, h_{124}$	[25.14.56]	$(0,1,-1,0,1)$
$\lambda\mu\rho = \nu\rho = 1$	$h_{35},\, h_{134}$	[35.14.56]	$(0,1,-1,0,-1)$
$\lambda = \rho = 1$	$h_{23},\, h_{34}$	[234.156]	$(0,1,1,1,0)$
$\lambda = \mu\nu\rho = 1$	$h_{23},\, h_{125}$	[23.15.46]	$(0,1,1,-1,0)$
$\mu\nu\rho = \lambda\rho = 1$	$h_{24},\, h_{125}$	[24.15.36]	$(0,1,-1,1,0)$
$\rho = \lambda\mu\nu\rho = 1$	$h_{34},\, h_{135}$	[34.15.26]	$(0,1,-1,-1,0)$

$-\epsilon_3 - \epsilon_4$	[145.236]	$(1,0,0,1,1)$
$-\epsilon_3 + \epsilon_4$	[14.23.56]	$(1,0,0,1,-1)$
$\epsilon_3 - \epsilon_4$	[15.23.46]	$(1,0,0,-1,1)$
$\epsilon_3 + \epsilon_4$	[45.23.16]	$(1,0,0,-1,-1)$
$-\epsilon_2 - \epsilon_4$	[135.246]	$(1,0,1,0,1)$
$-\epsilon_2 + \epsilon_4$	[13.24.56]	$(1,0,1,0,-1)$
$\epsilon_2 - \epsilon_4$	[15.24.36]	$(1,0,-1,0,1)$
$\epsilon_2 + \epsilon_4$	[35.24.16]	$(1,0,-1,0,-1)$
$-\epsilon_2 - \epsilon_3$	[134.256]	$(1,0,1,1,0)$
$-\epsilon_2 + \epsilon_3$	[13.25.46]	$(1,0,1,-1,0)$
$\epsilon_2 - \epsilon_3$	[14.25.36]	$(1,0,-1,1,0)$
$\epsilon_2 + \epsilon_3$	[34.25.16]	$(1,0,-1,-1,0)$

$-\epsilon_1 - \epsilon_4$	[125.346]	$(1,1,0,0,1)$
$-\epsilon_1 + \epsilon_4$	[12.34.56]	$(1,1,0,0,-1)$
$\epsilon_1 - \epsilon_4$	[15.34.26]	$(1,-1,0,0,1)$
$\epsilon_1 + \epsilon_4$	[25.34.16]	$(1,-1,0,0,-1)$
$-\epsilon_1 - \epsilon_3$	[124.356]	$(1,1,0,1,0)$
$-\epsilon_1 + \epsilon_3$	[12.35.46]	$(1,1,0,-1,0)$
$\epsilon_1 - \epsilon_3$	[14.35.26]	$(1,-1,0,1,0)$
$\epsilon_1 + \epsilon_3$	[24.35.16]	$(1,-1,0,-1,0)$
$-\epsilon_1 - \epsilon_2$	[123.456]	$(1,1,1,0,0)$
$-\epsilon_1 + \epsilon_2$	[12.45.36]	$(1,1,-1,0,0)$
$\epsilon_1 - \epsilon_2$	[13.45.26]	$(1,-1,1,0,0)$
$\epsilon_1 + \epsilon_2$	[23.45.16]	$(1,-1,-1,0,0)$

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