THE CAYLEY-OGUISO FREE AUTOMORPHISM OF POSITIVE ENTROPY ON A K3 SURFACE

DINO FESTI, ALICE GARBAGNATI, BERT VAN GEEMEN, AND RONALD VAN LUIJK

Abstract. Recently Oguiso showed the existence of K3 surfaces that admit a free automorphism of positive entropy. The K3 surfaces used by Oguiso have a particular rank two Picard lattice. We show, using results of Beauville, that these surfaces are therefore determinantal quartic surfaces. Long ago, Cayley constructed an automorphism of such determinantal surfaces. We show that Cayley's automorphism coincides with Oguiso's free automorphism. We also exhibit an explicit example of a determinantal quartic whose Picard lattice has exactly rank two and for which we thus have an explicit description of the automorphism.

Recently Keiji Oguiso showed that there exist projective K3 surfaces S with a fixed point free automorphism g of positive entropy, i.e. g^* has at least one eigenvalue λ of absolute value $|\lambda| > 1$ on $H^2(S,\mathbb{C})$ (see [O]). He also described the Picard lattice, of rank two, of the general such surface explicitly and observed that these surfaces can be embedded into $\mathbb{P}^3$ as a quartic surfaces. There remained the problem of describing these quartic surfaces and their automorphism g explicitly.

The aim of this paper is to provide a general method for constructing such quartic surfaces in $\mathbb{P}^3$ and to describe an algorithm for finding the automorphism. Moreover, we will give an explicit example of such a surface S and automorphism g. To identify the quartic surfaces in Oguiso's construction, we observe that the Picard lattice required by Oguiso is exactly the Picard lattice of a general determinantal quartic surface, that is, the quartic equation of the surface is the determinant of a 4×4 matrix of linear forms.

While writing the paper, we realised that such automorphisms were already described by Prof. Cayley, President of the London Mathematical Society, in his memoir on quartic surfaces, presented on February 10, 1870 ([C], §69, p.47). In fact, Cayley observed that a determinantal K3 surface $S_0 \subset \mathbb{P}^3$ has three embeddings $S_i \subset \mathbb{P}^3$ for $i = 0, 1, 2$, each of which is again determinantal. Moreover, the three matrices M_i provide natural (non-linear!) maps between these three quartic surfaces. A composition of these maps is an automorphism of S_0 and we show that this automorphism is the one discovered by Oguiso.

The three matrices involved are obtained from a single 'tritensor'. It would be interesting to know if the automorphism can be constructed directly from the tritensor and if there are generalizations to higher dimensions.

In the first section we recall Oguiso's description [O] of K3 surfaces with a fixed point free automorphism g of positive entropy. In section 1.6 we give a method which in principle allows one to give an explicit description of the automorphism. In practice, even if the K3 surface S

is given as a determinantal surface in $\mathbb{P}^3$, this method is hard to use, since one needs to know certain curves of high degree on S that are not complete intersections.

In the second section, using results of Beauville, we give a characterisation of the K3 surfaces considered by Oguiso as determinantal quartics. Given the matrix $M_0(x)$ whose determinant is a defining polynomial for S_0, Cayley indicated a method to find the corresponding matrices $M_1(y)$ and $M_2(z)$ for S_1 and S_2. We were not able to show that the determinants of M_1 and M_2 do not vanish identically in general. However, in the explicit example presented in Section 3, his method works and this allows us to give a convenient explicit description of the automorphism g in that case.

Acknowledgements. We are indebted to K. Oguiso and I. Dolgachev for helpful comments.

1. The general constructions

1.1. **The Lattice** (N, b)**.** To describe the Néron Severi group of K3 surfaces considered by Oguiso, we introduce a lattice (N, b). One has $N \cong \mathbb{Z}^2$, but to describe b and the isometries of (N, b) it is convenient to define

$$N := \mathbb{Z}[\eta], \qquad \eta^2 = 1 + \eta,$$

so the free $\mathbb{Z}$-module of rank two N is the ring of integers of $\mathbb{Q}(\eta) \cong \mathbb{Q}(\sqrt{5})$. The bilinear form is

$$b = b_N : N \times N \longrightarrow \mathbb{Z}, \qquad \text{with} \quad b(x, x) = 4(a^2 + ab - b^2)$$

where $x = a + b\eta$ with $a, b \in \mathbb{Z}$. More generally, $b(x, y) = (b(x + y, x + y) - b(x, x) - b(y, y))/2$, so we get

$$(N, b) \cong \left(\mathbb{Z}^2, \ S_b = S_{b_N} := \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \right).$$

For $x = a + b\eta \in N$ we denote its Galois conjugate by x'. Then one easily verifies that

$$b(x, y) = 2(x'y + xy') \qquad \text{with} \quad x' := a + b\eta', \quad \eta' := 1 - \eta.$$

As $\eta\eta' = -1$, the map

$$N \longrightarrow N, \qquad x = a + b\eta \longmapsto \eta^2 x = (a + b) + (a + 2b)\eta$$

is an isometry of the lattice (N, b) with inverse $\eta^{-2} = 2 - \eta$. Composing this map with itself n times gives an isometry which we denote simply by η^{2n}.

With these definitions, Oguiso proved the following theorem.

1.2. **Theorem ([O], Theorem 4.1).** There exist K3 surfaces S with $\mathrm{NS}(X) \cong (N, b)$. Moreover, any such surface S admits a free automorphism of positive entropy g such that $g^* = \eta^6$ on $\mathrm{NS}(S) \subset \mathrm{H}^2(S, \mathbb{C})$ and $g^* = -1$ on the orthogonal complement of $\mathrm{NS}(S)$ in $\mathrm{H}^2(S, \mathbb{C})$.

1.3. **Fibonacci numbers.** We will need to know the following values of $\eta^{2n} \in N$ explicitly:

$$\eta^2 = 1+\eta, \qquad \eta^4 = 2+3\eta, \qquad \eta^6 = 5+8\eta \ ,$$

as well as their inverses, with $\eta^{-2} = (\eta')^2 = 2-\eta$:

$$\eta^{-2} = 2-\eta, \qquad \eta^{-4} = 5-3\eta, \qquad \eta^{-6} = 13-8\eta \ .$$

The reader will notice the appearance of Fibonacci numbers ([O], Lemma 3.1): $\eta^{2n} = a_{2n-1} + a_{2n}\eta$ where $a_1 = a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$ and $\eta^{-2n} = a_{2n+1} - a_{2n}\eta$.

1.4. **Very ample and effective divisors on S.** Let S be a K3 surface with Picard lattice $\mathrm{NS}(S) \cong (N, b)$ as in section 1.1. We will fix the identification $\mathrm{NS}(S) \cong N$ in such a way that if D is an ample divisor class, so $D^2 > 0$, then $D = a + b\eta$ with $a > 0$. As there are no elements with $b(x,x) = -2$ in N, any $x = a + b\eta \in N$ with $b(x,x) > 0$ and $a > 0$ is the class of an ample divisor on S:

$$\mathcal{A}(S) = \left\{ x = a + b\eta \in \mathrm{NS}(S) : \ a > 0, \quad b(x,x) = 4(a^2+ab-b^2) > 0 \right\} .$$

As moreover $b(x,x) \neq 0$ for $x \neq 0$, results from [SD] imply that any ample divisor is already very ample.

Let $D \in \mathrm{NS}(S)$ be the class of an irreducible curve C, then by adjunction $D^2 = 2p_a(C) - 2 \geq -2$, and thus actually $D^2 > 0$. As also $DH > 0$ for any ample divisor H, we conclude that $D = a + b\eta$ with $a > 0$ and therefore any curve in S is an ample divisor. Taking linear combinations with positive coefficients of classes of curves, we conclude that any effective divisor on S is an ample divisor.

For any integer n we thus have the very ample divisor class

$$D_n := \eta^{2n} \in N, \qquad D_n^2 = D_0^2 = 4 \qquad (n \in \mathbb{Z}) \ .$$

The global sections of the corresponding line bundles on S define projective embeddings, denoted by ϕ_n, of S in $\mathbb{P}^3$ as a quartic surface $S_n \subset \mathbb{P}^3$:

$$\phi_n := \phi_{D_n} : S \xrightarrow{\cong} S_n \subset \mathbb{P}^3 \ .$$

1.5. **The automorphism g.** The remarkable fact that there is an automorphism of S with $g^* = \eta^6$ implies that the quartic surfaces S_0 and S_3 are the same, after choosing suitable coordinates on the $\mathbb{P}^3$'s.

In fact, let $s_0, \ldots, s_3$ be a basis of $\mathrm{H}^0(S, D_0)$, then as $g^*D_0 = D_3$, $\mathrm{H}^0(S, D_3)$ has basis $t_i := g^*s_i$, where $i = 0, \ldots, 3$. With a slight abuse of notation, we thus get for all $x \in S$:

$$\phi_3(x) = (t_0(x) : \ldots : t_3(x)) = (s_0(g(x)) : \ldots : s_3(g(x))) = \phi_0(g(x)) \ .$$

Thus, with these bases, $S_3 = S_0 \subset \mathbb{P}^3$. Moreover $\phi_3 = \phi_0 \circ g$ implies that

$$g = \phi_0^{-1} \circ \phi_3 : S \longrightarrow S \ .$$

1.6. **How to find g.** To give a more concrete description of g, we explain how, in principle, one can describe ϕ_3 in terms of ϕ_0. For this we need to find $\mathrm{H}^0(S, D_3)$, given the surface $S_0 \subset \mathbb{P}^3$. The zero locus of a global section t of D_3 is mapped to curve in S_0. This curve is not the (complete) intersection of S_0 with another surface (of degree d) in $\mathbb{P}^3$, since such an intersection has class $dD_0 = d$, whereas $D_3 = \eta^6 = 5 + 8\eta$.

Notice that $\eta^{-6} = 13 - 8\eta$ and thus $\eta^6 + \eta^{-6} = 18$, so $D_3 + D_{-3} = 18D_0$. Let $C_i := (t_i = 0)$ be the zero divisors of a basis t_i, $i = 0, \ldots, 3$, of $\mathrm{H}^0(S, D_3)$ and similarly, let the t'_j be a basis of $\mathrm{H}^0(S, D_{-3})$ with zero divisor $C'_j := (t'_j = 0)$. Then the divisor $C_i + C'_j$ has class $D_3 + D_{-3} = 18D_0$ and it is the zero locus of the section $t_i t'_j$ in $\mathrm{H}^0(S, 18D_0)$.

Consider the exact sequence of sheaves on $\mathbb{P}^3$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \longrightarrow i_*\mathcal{O}_{S_0}(d) \longrightarrow 0 ,$$

where the first non-trivial map is multiplication by the equation of S_0 and where $i : S_0 \hookrightarrow \mathbb{P}^3$ is the inclusion map. As H^1 of any invertible sheaf on $\mathbb{P}^3$ is zero, and $\phi_0^*\mathcal{O}_{S_0}(d) = dD_0$, we get a surjection

$$\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \xrightarrow{\phi_0^*} \mathrm{H}^0(S, dD_0) \longrightarrow 0 .$$

Therefore, for any d, a section in $\mathrm{H}^0(S, dD_0)$ is the restriction of a homogeneous polynomial of degree d on $\mathbb{P}^3$.

In particular, there are homogeneous polynomials R_{ij} of degree 18 in $x_0, \ldots, x_3$, such that

$$\phi_0^* R_{ij} = t_i t'_j \quad \in \mathrm{H}^0(S, 18D_0) , \qquad (i, j \in \{0, \ldots, 3\}) .$$

Considering the zero loci of these sections we get:

$$(R_{ij} = 0) \cap S_0 = \phi_0(C_i) + \phi_0(C'_j) .$$

The curves $\phi_0(C_i)$, $\phi_0(C'_j)$ in $\mathbb{P}^3$ both have degree $D_0 C_i = 36 = D_0 C'_j$, consistent with $18 \cdot 4 = 72 = 36 + 36$.

The map ϕ_3 is defined by the global sections $t_0, \ldots, t_3$ of D_3. Since D_{-3} is very ample, for each $x \in S$ there is an index j such that $t'_j(x) \neq 0$. So (with slight abuse of notation):

$$\begin{aligned} \phi_3 : S \longrightarrow S_3 \subset \mathbb{P}^3, \qquad p \longmapsto \quad & (t_0(p) : \ \ldots \ : t_3(p)) \\ = \quad & (t_0(p)t'_j(p) : \ \ldots \ : t_3(p)t'_j(p)) \\ = \quad & (R_{0j}(\phi_0(p)) : \ldots : R_{3j}(\phi_0(p))) . \end{aligned}$$

On the open subset of S where $t'_j \neq 0$, we thus have: $\phi_3 = R_j \circ \phi_0$, where $R_j : \mathbb{P}^3 \to \mathbb{P}^3$ is the rational map given by the polynomials $R_{0j}, \ldots, R_{3j}$. Hence on this open subset we get

$$g = \phi_0^{-1} \circ \phi_3 = \phi_0^{-1} \circ R_j \circ \phi_0 ,$$

that is, if we identify S with S_0, then g is just the rational map R_j, for any j, and these maps glue to give an isomorphism $S_0 \to S_0$, which 'is' g.

To find ϕ_3, given ϕ_0, we thus need to find the polynomials R_{ij} on $\mathbb{P}^3$. In practice, even if one is given that $\mathrm{NS}(S) \cong (N, b)$ this seems quite difficult. Fortunately, there is simpler way to proceed as we will see in the Section 2.

1.7. **Moving from S_n to S_{n+1}.** We just explained how to move from S_0 to S_3. It will useful to have a description of how to move from S_n to S_{n+1}, assuming that we know how to move from S_{n-1} to S_n. The starting point for these steps will be provided by a characterization of the quartic surfaces S with $\mathrm{NS}(S) \cong (N, b)$ in Section 2.1.

The identity

$$\eta^4 - 3\eta^2 + 1 = 0 \quad \text{implies} \quad \eta^{2n+2} - 3\eta^{2n} + \eta^{2n-2} = 0, \quad \text{hence} \quad 3D_n = D_{n-1} + D_{n+1}.$$

Following the procedure outlined in the previous section, we find that if we have a curve $C' \subset S$ with class D_{n-1}, then the cubics that vanish on $\phi_n(C')$ define, on the open subset $S_n - \phi_n(C')$ of S_n, the map $\phi_{n+1} \circ \phi_n^{-1} : S_n \to S_{n+1} \subset \mathbb{P}^3$.

2. Cayley's description of the automorphism

2.1. **Determinantal quartics.** We now show that a result of Beauville provides an explicit description of the K3 surfaces we are interested in: the K3 surfaces S with Néron Severi group isomorphic to (N, b) are exactly the quartic determinantal surfaces with Picard number two.

As observed by Cayley, given a quartic determinantal surface, it is easy to find two others and to find isomorphisms between them. In our setup, starting from the projective model S_0 of S, he produces S_1 and S_2 and the composition of the isomorphisms $S_0 \to S_1 \to S_2 \to S_3 = S_0$ is basically the automorphism g we wanted to describe.

2.2. **Proposition.** Let S be a K3 surface with Néron Severi group $\mathrm{NS}(S) \cong (N, b)$ as in Section 1.1. Then, for any $n \in \mathbb{Z}$, the quartic surface $S_n := \phi_n(S)$ is determinantal. So there is a 4×4 matrix $M_n(x)$, whose coefficients are linear forms in 4 variables $x_0, \ldots, x_3$, such that $\det M_n(x) = 0$ is an equation for S_n.

Conversely, a general determinantal quartic surface S has $\mathrm{NS}(S) \cong (N, b)$ and thus it admits a fixed point free automorphism of positive entropy.

Proof. The proposition is an easy consequence of [B], Proposition 6.2, where Beauville proved that a smooth quartic surface X is determinantal if and only if there is a curve $C \subset X$ of degree 6 and genus 3. See also [D], Section 4.2.5.

Given a K3 surface S with $\mathrm{NS}(S) \cong (N, b)$, there are smooth genus three curves C_n on S with class the very ample divisor $D_n = \eta^{2n}$, for all $n \in \mathbb{Z}$ (cf. Section 1.4). As multiplication by η^{-2n} is an isometry of the lattice (N, b), and as one easily computes $C_0 C_1 = 6$, we then get that $C_n C_{n+1} = 6$. As $\phi_n(C_n)$ is a plane section of S_n, the curve $\phi_n(C_{n+1})$ is a smooth genus three curve of degree 6 in S_n. Hence S is determinantal.

For the converse, let S be a general determinantal quartic surface in $\mathbb{P}^3$. Then S is smooth. Let H be the hyperplane class of S, so $H^2 = 4$. Let $C \subset S$ be a degree 6 and genus 3 as in

[B] Proposition 6.2. Then $HC = 6$ and the adjunction formula implies that $C^2 = 4$. Thus the intersection form on the sublattice $\mathbb{Z}H \oplus \mathbb{Z}C$ of $\mathrm{NS}(S)$ is given by the matrix

$$\begin{pmatrix} H^2 & HC \\ HC & C^2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} .$$

This sublattice is isometric to (N, b) since the $\mathbb{Z}$-basis of N given by $D_0 = (1,0)$ and $D_1 = (1,1)$ gives this intersection matrix. Thus $\mathrm{NS}(S)$ of a determinantal quartic K3 surface contains (N, b) as a sublattice. A count of parameters shows that determinantal quartics have 18 moduli, and thus the rank of the Néron Severi group is 2 for a general determinantal quartic. Alternatively, in the next section we provide an example of a (smooth) determinantal quartic with $\mathrm{rank}(\mathrm{NS}(S)) = 2$, thus the same is true for the general determinantal quartic.

For such a quartic we thus have $\mathrm{NS}(S) \subset N$, of finite index. As $|\det(b)| = 20$, the index can only be 1 or 2. If the index is two then $D := (aH + bC)/2 \in \mathrm{NS}(S)$ with $(a, b) = (1, 0)$ or $(0, 1)$ or $(1, 1)$, but D^2 is odd in all these cases, so this is impossible. Hence $\mathrm{NS}(S) = (N, b)$ for a general determinantal quartic surface. The existence of a fixed point free automorphism of positive entropy now follows from Oguiso's results in [O]. $\square$

2.3. **Generators of** $\mathrm{NS}(S)$**.** The proposition implies in particular that a general smooth determinantal surface $S \subset \mathbb{P}^3$ has Néron Severi group of rank two. One would thus like to see a curve on S which is not a complete intersection, that is, whose class is not an integer multiple of the hyperplane class H of $S \subset \mathbb{P}^3$. As explained in [B] (see also [D], Example 4.2.4), such curves, of genus 3 and degree 6, are easy to find as follows.

The matrix of linear forms M, whose determinant defines S, also gives a sheaf homomorphism $\mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus 4}$ on $\mathbb{P}^3$. The cokernel is $i_*\mathcal{L}$ for an invertible sheaf $\mathcal{L}$ on S, where $i : S \hookrightarrow \mathbb{P}^3$ is the inclusion ([B], Corollary 1.8).

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus 4} \xrightarrow{M} \mathcal{O}^{\oplus 4} \longrightarrow i_*\mathcal{L} \longrightarrow 0 .$$

So M defines a line bundle on S with sheaf of sections $\mathcal{L}$. As $\mathrm{H}^i(\mathbb{P}^3, \mathcal{O}(-1)) = 0$ for all i, we obtain an isomorphism

$$\mathbb{C}^4 = \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}^{\oplus 4}) \xrightarrow{\cong} \mathrm{H}^0(S, \mathcal{L}) .$$

2.4. **Proposition.** ([B], (6.7)) Let $s \in \mathrm{H}^0(S, \mathcal{L})$ be the global section of $\mathcal{L}$ which is the image of the first basis vector of $\mathbb{C}^4$ and let C be the zero locus of s.

Then $C \subset S$ is the curve defined by the vanishing of all maximal minors of the 3×4 matrix obtained from M by deleting the first row. Moreover, C is a curve of degree 6 and genus 3.

Proof. Notice that $s(x) = 0$ if and only if $e(x) \in \mathrm{im}M(x)$ where e is the global section of the trivial bundle $\mathcal{O}^{\oplus 4}$ defined by the first basis vector. Applying the cofactor matrix $P(x)$ of $M(x)$ to the identity $e(x) = M(x)v$, for some $v = v(x) \in \mathbb{C}^4$, we get $P(x)e(x) = (\det M(x))v = 0$ for

$x \in S$. As $P(x)e(x)$ is the vector with components $P_{i1}(x)$, and as these coefficients of $P(x)$ are, up to sign, the maximal minors as in the proposition, we get the first part of the proposition.

The degree and genus of the curve are given in [B] Prop. 6.2, [D] Thm. 4.12.14. The genus is actually easy to compute in this case: as $\mathcal{L} \cong \mathcal{O}_S(C)$ and all effective divisors on S are ample, Kodaira vanishing and Riemann-Roch on S imply that $4 = \dim \mathrm{H}^0(S, \mathcal{O}_S(C)) = \chi(\mathcal{O}_S(C)) = p_a(C) + 1$. □

2.5. **Towards explicitly moving from S_n to S_{n+1}.** Given a determinantal surface S with equation $\det M = 0$, one obviously also has the (same) equation $\det {}^tM = 0$. However, the invertible sheaf $\mathcal{L}'$ on S defined by the cokernel of ${}^tM$ is not isomorphic to $\mathcal{L}$, but to $\mathcal{O}_S(3) \otimes \mathcal{L}^{-1}$ ([B], (6.3), [D], (4.19)). The maximal minors of M and ${}^tM$ are however the same, but rows and columns are of course interchanged.

Now we return the determinantal surfaces S_n. The following corollary identifies the class of the curve C in $\mathrm{NS}(S_n)$ in the proposition up to this ambiguity. It is exactly what we need to be able to apply the method we outlined in Section 1.7 to move from S_n to S_{n+1}.

2.6. **Corollary.** Let S be a K3 surface with Néron Severi group $\mathrm{NS}(S) \cong (N, b)$ as in Section 1.1. Let $S_n = \phi_n(S) \subset \mathbb{P}^3$ be the smooth determinantal surface defined by $\det M_n = 0$. Let $P_{n,ij}$ be the cubic polynomial in 4 variables which is the determinant of the 3×3 submatrix of M_n obtained by deleting the i-th row and j-th column.

Then the cubic surface $P_{n,ij} = 0$ intersects S_n in the union of two curves

$$\big(P_{n,ij} = 0\big) \cap S_n \,=\, C_{n,i} \,+\, C'_{n,j}\ ,$$

each of degree 6 and genus 3, with classes D_{n+1}, D_{n-1} respectively.

Proof. To find the classes of the curves, we observe that if $x \in \mathcal{A}$ and $b(x, x) = 4$ then $xx' = 1$, so x is a unit in the ring $\mathbb{Z}[\eta]$. The group of units of $\mathbb{Z}[\eta]$ is well-known to be isomorphic to be the set $\{\pm\eta^m : m \in \mathbb{Z}\}$. As $b(\eta, \eta) = 4\eta\eta' = -4$, we have $x = \eta^m$ with m even (the sign is $+$ since $x \in \mathcal{A}$). Thus each of the two curves has a class D_m for some m and the sum of the classes must be $3D_n$.

Now we use that the hyperplane class of S_n is $D_n = \eta^{2n}$ and that multiplication by η^{-2n} is an isometry of N. As the curves have degree 6, we must have $D_m D_n = 6$, equivalently, $D_k D_0 = 6$, with $k = m - n$. It is easy to check that there are only two D_k's with this property, $k = \pm 1$. As the sum of the D_k must be $3D_0$, one must be D_1 and the other D_{-1}. Thus $\big(P_{n,ij} = 0\big) \cap S_n$ is the union of two curves, with classes D_{n+1} and D_{n-1} respectively.

After replacing M by ${}^tM$, Proposition 2.4 shows that the intersection of all four cubic surfaces $(\cap_i P_{n,i1} = 0)$ with S_n is a curve, which we denote by $C'_{n,1}$, with class D_{n-1}. Thus $\big(P_{n,i1} = 0\big) \cap S = C_{n,i} + C'_{n,1}$ for some curves $C'_{n,i}$ with class D_{n+1}. Repeating this with $\cap_i (P_{n,ij} = 0)$, $j = 1, 2, 3, 4$, the corollary follows. □

2.7. **Explicitly moving form S_n to S_{n+1}.** The cubic polynomials $P_{n,ij}$ defined in Corollary 2.6, for $i = 1, \ldots, 4$ and fixed j, all vanish on $\phi_n(C'_j)$. From the description of the map $\phi_{n+1}\phi_n^{-1}$ given in Section 1.7 we thus obtain that this map is given by

$$\phi_{n+1}\phi_n^{-1} : S_n \longrightarrow S_{n+1}, \qquad x \longmapsto (P_{n,1j}(x) : \ldots : P_{n,4j}) .$$

Thus if we know a determinantal representation $\det M_n(x) = 0$ of S_n, the maximal minors of $M_n(x)$ define the map from S_n to S_{n+1}.

To keep moving, one now needs to find a determinantal representation of S_{n+1}. It was observed by Cayley that the matrix $M_n(x)$ defines another matrix, whose determinant, if not identically zero, will define S_{n+1}. Thus, in favorable circumstances, to find all the matrices $M_i(x)$ whose determinants define the S_i, and whose cofactor matrices define the isomorphisms $S_i \to S_{i+1}$, it suffices to know the matrix $M_0(x)$. This is actually what happens in the explicit example provided in Section 3, and it allows one to find explicit polynomials defining $g : S = S_0 \to S_3 = S$. Therefore we reproduce Cayley's remarks below.

2.8. **Cayley's description of the automorphism g.** Let S be a K3 surface with Néron Severi group (N, b) and let $S_0 = \phi_0(S) \subset \mathbb{P}^3$. From Proposition 2.2 we know that S_0 is determinantal, thus there is a 4×4 matrix $M_0(x)$,

$$M_0(x) := \big(m_{kj}(x)\big)_{k,j=0,\ldots,3} , \qquad \text{with} \quad m_{kj}(x) := \sum_{i=0}^{3} a_{ijk}x_i ,$$

and $a_{ijk} \in \mathbb{C}$ such that

$$S_0 : \quad \det M_0(x) = 0 \qquad (\subset \mathbb{P}^3) .$$

The $4^3 = 64$ complex numbers a_{ijk} can be viewed as the components of a 'tritensor' in $(\mathbb{C}^4)^{\otimes 3}$. There are three obvious ways in which this tritensor defines a 4×4 matrix of linear forms. These will be the matrices $M_0(x), M_1(y), M_2(z)$ below whose determinants define S_1, S_2 and S_3 respectively, provided these determinants are not identically zero.

Let $P_0(x)$ be the cofactor matrix of $M_0(x)$, so

$$P_0(x)M_0(x) = M_0(x)P_0(x) = (\det M_0(x))I,$$

where I is the 4×4 identity matrix. For $x \in S_0$, we have $\det M_0(x) = 0$ and thus each column of $P_0(x)$, provided it is not identically zero, provides a non-trivial solution to the linear equations $M_0(x)y = 0$. As S_0 is smooth, the rank of $M_0(x)$ is equal to three for any $x \in S_0$ and thus y is unique up to scalar multiple.

On the other hand, each column of $P_0(x)$ provides a point of S_1 (or S_{-1}, but in that case we replace M_0 by its transpose). As $\det M_0(x) = 0$ exactly for $x \in S_0$, we get, with some abuse of notation,

$$S_1 = \{y \in \mathbb{P}^3 : \exists x \in \mathbb{P}^3 \quad \text{s.t.} \quad M_0(x)y = 0\} .$$

The system of linear equations $M_0(x)y = 0$ can be rewritten as:

$$0 = \sum_{j=0}^{3} \big(\sum_{i=0}^{3} a_{ijk}x_i\big)y_j = \sum_{i,j=0}^{3} a_{ijk}x_iy_j = \sum_{i=0}^{3} x_i\big(\sum_{j=0}^{3} a_{ijk}y_j\big), \qquad (k = 0, \dots, 3) .$$

This set of four equations is equivalent to the matrix equation,

$${}^t x M_1(y) = 0, \qquad M_1(y) := \big(m'_{ik}(y)\big), \quad m'_{ik}(y) := \sum_{j=0}^{3} a_{ijk}y_j .$$

For $y \in \mathbb{P}^3$ these equations have a non-trivial solution $x = x(y)$ if and only if $\det M_1(y) = 0$. In case $\det M_1(y)$ is not identically zero, it is a quartic polynomial that vanishes on the quartic surface S_1, and thus it is a defining equation for S_1.

We will *assume* that $\det M_1(y)$ is not identically zero. Then

$$S_1 : \quad \det M_1(y) = 0 \qquad (\subset \mathbb{P}^3) ,$$

and we can repeat this procedure: let $P_1(y)$ be the cofactor matrix of $M_1(y)$. As S_0 consists of the points x with ${}^t x M_1(y) = 0$ and $P_1(y)M_1(y) = 0$ for $y \in S_1$, each row of P_1 defines the map $S_1 \to S_0$. Thus the columns of P_1 define the map to S_2. So for $y \in S_1$, as each column of $P_1(y)$ is both a solution z of $P_1(y)z = 0$ and defines a point of S_2, we have

$$S_2 = \{z \in \mathbb{P}^3 : \exists y \in \mathbb{P}^3 \quad \text{s.t.} \quad M_1(y)z = 0\} .$$

Rewriting the linear equations $M_1(y)z = 0$, we get ${}^t y M_2(z) = 0$ with

$$M_2(z) := \big(m''_{ji}(z)\big), \quad m''_{ji}(z) := \sum_{k=0}^{3} a_{ijk}z_k .$$

Now, *assuming moreover* that $\det M_2(z)$ is not identically zero, we get

$$S_2 : \quad \det M_2(z) = 0 \qquad (\subset \mathbb{P}^3) .$$

Finally we consider the cofactor matrix $P_2(z)$ of $M_2(z)$. The columns of $P_2(z)$ provide us with the map $S_2 \to S_3$ and they are solutions of $M_2(z)x = 0$. Rewriting this system, we get ${}^t z M_0(x) = 0$, showing that S_3 is defined by $\det M_0(x) = 0$ since this determinant *is* not identically zero, being the defining equation of S_0.

Thus $S_0 = S_3(!)$ and the composition of the maps, each given by cubic polynomials (the minors of the matrices M_i)

$$S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow S_0$$

is $\phi_0 \circ g \circ \phi_0^{-1}$, where the automorphism g constructed by Oguiso. To quote Cayley: “The process may be indefinitely repeated”.

3. An explicit example

3.1. **The method.** In this section we give an explicit example of a determinantal K3 surface with Picard number two. The main problem is to give an upper bound for the Picard number. For this we use a method described in [vL].

The following result shows that if a smooth projective surface X over a number field K has good reduction at a prime $\mathfrak{p}$, then the geometric Picard number of X is bounded from above by the geometric Picard number of the reduction.

3.2. **Proposition.** Let K be a number field with ring of integers $\mathcal{O}$, let $\mathfrak{p}$ be a prime of $\mathcal{O}$ with residue field k, and let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$. Let $\mathfrak{X}$ be a smooth projective surface over $\mathcal{O}_\mathfrak{p}$ and set $X_{\overline{K}} = \mathfrak{X} \times_{\mathcal{O}_\mathfrak{p}} \overline{K}$ and $X_{\overline{k}} = \mathfrak{X} \times_{\mathcal{O}_\mathfrak{p}} \overline{k}$. Let ℓ be a prime not dividing $q = \#k$. Let Ψ denote the automorphism of $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1))$ induced by the q-th power Frobenius $F_q \in \mathrm{Gal}(\overline{k}/k)$.
Then there are natural injections

$$\mathrm{NS}(X_{\overline{K}}) \otimes_\mathbb{Z} \mathbb{Q}_\ell \hookrightarrow \mathrm{NS}(X_{\overline{k}}) \otimes_\mathbb{Z} \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1)) ,$$

that respect the intersection pairing and the action of Frobenius, respectively. The rank of $\mathrm{NS}(X_{\overline{k}})$ is at most the number of eigenvalues of Ψ that are roots of unity, counted with multiplicity.

Proof. See Proposition 6.2 and Corollary 6.4 in [vL1]. □

3.3. **An explicit determinant.** Consider the following matrix, whose entries are linear forms in the variables $x_0, \dots, x_3$ with integer coefficients:

$$N = \begin{pmatrix} x_0 & x_2 & x_1 + x_2 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_3 \\ x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\ x_0 + x_1 + x_3 & x_0 + x_2 & x_3 & x_2 \end{pmatrix} .$$

The following theorem shows that any such matrix that is congruent to N modulo 2 defines a determinantal quartic surface in $\mathbb{P}^3 = \mathbb{P}^3(\mathbb{C})$ with Picard number 2.

3.4. **Theorem.** Let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any matrix whose entries are homogeneous polynomials of degree 1 and such that M is congruent modulo 2 to the matrix N above. Denote by S the complex surface in $\mathbb{P}^3$ given by $\det M = 0$. Then S is a K3 surface and its Picard number equals 2.

Proof. Let $\mathfrak{S}$ denote the surface over the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at the prime 2 given by $\det M = 0$, and write S_2 and $\overline{S}_2$ for $\mathfrak{S}_{\mathbb{F}_2}$ and $\mathfrak{S}_{\overline{\mathbb{F}}_2}$, respectively. One easily checks that S_2 is smooth. Since smoothness is an open property, also $\mathfrak{S}$ is smooth and thus $S = \mathfrak{S}_\mathbb{C}$ is smooth, so it is a K3 surface. Let Ψ denote the automorphism of $\mathrm{H}^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))$ induced by Frobenius

$F_2 \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$. As $\mathrm{NS}(S) \cong \mathrm{NS}(\mathfrak{S}_{\bar{\mathbb{Q}}})$, it is enough to show that only two of the eigenvalues of Ψ are roots of unity.

Since $\overline{S}_2$ is a K3 surface, we have $\dim \mathrm{H}^i_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1)) = 1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$, respectively. The divisor classes in $\mathrm{H}^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))$ defined by the hyperplane class and the curve C as in Proposition 2.4 span a two-dimensional subspace V on which Ψ acts as the identity. We denote the linear map induced by Ψ on the quotient $W := \mathrm{H}^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))/V$ by Φ, so that $\mathrm{Tr}\,\Psi^n = \mathrm{Tr}(\Psi^n|_V) + \mathrm{Tr}\,\Phi^n = 2 + \mathrm{Tr}\,\Phi^n$ for every integer n. We denote the characteristic polynomial of a linear operator T by f_T, so that

$$f_\Psi = f_{\Psi|_V} \cdot f_\Phi = (x-1)^2 f_\Phi \ .$$

Using that Ψ is the inverse of the geometric Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))$, we may compute the traces of (powers of) Φ and Ψ with the Lefschetz trace formula, which, given that $\mathrm{H}^1 = \mathrm{H}^3 = 0$, yields

$$\#S_2(\mathbb{F}_{2^n}) \ = \ \sum_{i=0}^{2} 2^{ni} \cdot \mathrm{Tr}(F_2^n(i)) \ ,$$

where $\mathrm{Tr}(F_2^n(i))$ denotes the trace of the n-th power of the automorphism $F_2(i)$ of $\mathrm{H}^{2i}_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(i))$ induced by $F_2 \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$. In particular, $F_2(0)$ and $F_2(2)$ are the identity and $F_2(1) = \Psi$, so we obtain

$$\#S_2(\mathbb{F}_{2^n}) \ = \ 1 + 2^n \cdot \big(2 + \mathrm{Tr}(\Phi^n)\big) + 2^{2n} \ ,$$

from which it follows that

$$\mathrm{Tr}(\Phi^n) \ = \ -2 \ + \ \frac{\#S_2(\mathbb{F}_{2^n}) - 1 - 2^{2n}}{2^n} \ .$$

If $\lambda_1, \ldots, \lambda_{20}$ denote the eigenvalues of Φ, then the trace of the n-th power of Φ is given by

$$\mathrm{Tr}(\Phi^n) \ = \ \lambda_1^n + \ldots + \lambda_{20}^n \ ,$$

i.e. it is the n-th power sum symmetric polynomial in the eigenvalues of Φ.

We counted the number of points in $S_2(\mathbb{F}_{2^n})$ for $n = 1, \ldots, 10$ with Magma [M]. The results are in the table below. Knowing the values of the n-th power sum symmetric polynomial in the eigenvalues of Φ for $n = 1, \ldots, 10$, using Newton's identities, we computed the values e_n of the elementary symmetric polynomial of degree n in the eigenvalues of Φ for $n = 1, \ldots, 10$. These are the e_n in the table.

n	1	2	3	4	5	6	7	8	9	10
$\#S_2(\mathbb{F}_{2^n})$	6	26	90	258	1146	4178	17002	64962	260442	1044786
$e_n = (-1)^n c_n$	$-\frac{3}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	-1	2

The elementary symmetric polynomials are, up to sign, the coefficients of the characteristic polynomial f_Φ, hence we have computed the first ten coefficients of

$$f_\Phi = x^{20} + c_1 x^{19} + \ldots + c_{20} \; .$$

By Poincaré duality, f_Ψ satisfies the functional equation

$$x^{22} f_\Psi\Big(\frac{1}{x}\Big) = \pm f_\Psi(x) \; ,$$

i.e. f_Ψ, and thus f_Φ, is either palindromic or antipalindromic. As $c_{10} = 2 \neq 0$, we deduce that f_Φ is palindromic, and hence we know all its coefficients and find

$$f_\Psi = (x-1)^2 f_\Phi = (x-1)^2 \left(x^{20} + \tfrac{3}{2}x^{19} + x^{18} - \tfrac{1}{2}x^{13} + x^{11} + 2x^{10} + x^9 - \tfrac{1}{2}x^7 + x^2 + \tfrac{3}{2}x + 1\right).$$

Magma verified that f_Φ is irreducible. As it does not have integer coefficients, its roots are not algebraic integers, hence none of its roots is a root of unity.

This implies that Ψ acting on $\mathrm{H}^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_l(1))$ has only two eigenvalues (counted with multiplicity) that are roots of unity, and so, by Proposition 3.2, it follows that the rank of the Picard group of S is bounded by two from above. On the other hand, by Proposition 2.2 we know that the rank is at least two, hence $\mathrm{NS}(S)$ has rank two. □

3.5. **The automorphism.** We tried to apply Cayley's method, starting with the choice $M_0(x) := N$ as in Section 3.3. We found that the successive matrices $M_1(y)$ and $M_2(z)$ as in Section 2.8 have determinants which are not identically zero. Cayley's method therefore gives the free automorphism g explicitly as the composition of three maps, each given by cubic polynomials. Each coordinate function of g is thus homogeneous of degree $3^3 = 27$ on $\mathbb{P}^3$.

The automorphism g can given by polynomials R_{ij} of degree 18, as we showed in section 1.6. In order to find such polynomials, we tried to use Magma to find irreducible components of the intersection of $S = S_0$ with the zero locus of a degree 27 coordinate function, but we did not succeed.

References

[B] A. Beauville, *Determinantal Hypersurfaces*, Michigan Math. J. **48** (2000) 39–64.

[C] A. Cayley, *A memoir on quartic surfaces*, Proc. London Math. Soc. **3** (1869-71) 19–69.

[D] I. Dolgachev, Classical Algebraic Geometry: a modern view. To appear, Cambridge University Press.

[vL1] R. van Luijk, *An elliptic K3 surface associated to Heron triangles*, J. Number Theory **123** (2007), 92–119.

[vL] R. van Luijk, *K3 surfaces with Picard number one and infinitely many rational points*, Algebra and Number Theory **1**, No. **1** (2007), 1–15.

[M] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997) 235–265.

[O] K. Oguiso, *Free automorphisms of positive entropy on smooth Kähler surfaces*, arXiv:1202.2637v3, to appear in Adv. Stud. Pure Math.

[SD] B. Saint-Donat, *Projective models of K3 surfaces*, Amer. J. Math. **96** (1974) 602–639.

Matematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, 2333 Leiden, Nederland

Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italia

Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italia

Matematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, 2333 Leiden, Nederland
E-mail address: dinofesti@hotmail.it
E-mail address: alice.garbagnati@unimi.it
E-mail address: lambertus.vangeemen@unimi.it
E-mail address: rmluijk@gmail.com