A REMARK ON GENERALIZED COMPLETE INTERSECTIONS

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ABSTRACT. We observe that an interesting method to produce non-complete intersection sub-varieties, the generalized complete intersections from L. Anderson and coworkers, can be understood and made explicit by using standard Cech cohomology machinery. We include a worked example of a generalized complete intersection Calabi-Yau threefold.

INTRODUCTION

Calabi-Yau varieties, in particular those of dimension three, are of great interest in string theory. Since there are not many general results yet on their classification, but see [W], the explicit construction of CY threefolds is a quite important enterprise. For example, Kreuzer and Starke classified the toric fourfolds which have CY threefolds as (anticanonical) hypersurfaces [KS], [AGHJN]. Besides generalizations to complete intersection CYs in certain ambient toric varieties, like products of projective spaces, there are various other examples of CY threefolds constructed with more sophisticated algebro-geometrical methods. Recent examples include [IMOU], [CGKK], [KK].

In the recent paper [AAGGL], L. Anderson, F. Apruzzi, X. Gao, J. Gray and S-J. Lee found a very nice method to construct many more CY threefolds. The basic idea is to take a hypersurface Y in an ambient variety P and to consider hypersurfaces X in Y. These hypersurfaces need not be complete intersections in P, that is, there need not exist two sections of two line bundles on P whose common zero locus is X. There are various generalizations of this method, but we will stick to this basis case. As in [AAGGL], we refer to these varieties as generalized complete intersections (gCIs).

A particularly interesting and accessible case that was found and studied by Anderson and coworkers is when the ambient variety is a product of two varieties, one of which is $\mathbf{P}^1$, so $P = P_2 \times \mathbf{P}^1$. The variety P_2 they consider is a product of projective spaces, but this is not essential, one could consider any toric variety or even more general cases. The factor $\mathbf{P}^1$ is important since there are line bundles on $\mathbf{P}^1$ with non-trivial first cohomology group and this is essential to find generalized complete intersections. We review this construction in Section 1.1.

We provide a proposition, proven with standard Cech cohomology methods, that allows one, under a certain hypothesis, to find three equations (more precisely, three sections of three line bundles on P) that define X. In Section 2 we work out a detailed example, with explicit equations, of a CY threefold which was already considered in [AAGGL]. The explicit example X has an automorphism of order two and the quotient of X by the involution provides, after desingularization, another CY threefold. More generally, we think that among the gCIs found in [AAGGL] one could find more examples of CY threefolds with non-trivial automorphisms. It might be hard though to implement a systematic search as was done in [CGL] for complete intersection CY threefolds in products of projective spaces. We did not find new CY threefolds

with small Hodge numbers (see [CCM] for an update on these), but the gCICY seem to be a promising class of CYs to search for these. The recent paper [BH] by Berglund and Hübsch provides further techniques to deal with gCICYs whereas [AAGGL2] explores string theoretical aspects of gCICYs.

1. The construction of generalized complete intersections

1.1. **The general setting.** Let P_2 be a projective variety of dimension n and let $P := P_2 \times \mathbf{P}^1$. We denote the projections to the factors of P by π_1, π_2 respectively. For a coherent sheaf $\mathcal{F}$ on P_2 and an integer d we define a coherent sheaf on P by:

$$\mathcal{F}[d] := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{O}_{\mathbf{P}^1}(d) .$$

The Künneth formula gives

$$H^r(P, \mathcal{F}[d]) = \oplus_{p+q=r} H^p(P_2, \mathcal{F}) \otimes H^q(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) .$$

Recall that the only non-zero cohomology of $\mathcal{O}_{\mathbf{P}^1}(d)$ is: $h^0(\mathcal{O}_{\mathbf{P}^1}(d)) = h^1(\mathcal{O}_{\mathbf{P}^1}(-2-d)) = d+1$ for $d \geq 0$ and a basis for $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ is given by the monomials $z_0^i z_1^{d-i}$, $i = 0, \ldots, d$, where $(z_0 : z_1)$ are the homogeneous coordinates on $\mathbf{P}^1$.

Let L be a line bundle on P_2 and assume that $L[d]$, for some $d \geq 1$, has a non-trivial global section F. Using the Künneth formula, we can write $F = \sum_i f_i z_0^i z_1^{d-i}$ for certain sections $f_i \in H^0(P_2, L)$. Let $Y = (F)$ be the zero locus of F in P. We assume that Y is a (reduced, irreducible) variety, although this will not be essential in this section.

To define a codimension two subvariety of P, we consider another line bundle M on P_2. The Künneth formula shows that $M[-e]$ has no global sections if $e \geq 1$. But upon restricting to Y, the vector space $H^0(Y, M[-e]_{|Y})$ could still be non-trivial. In fact, from the exact sequence

$$(1) \qquad 0 \longrightarrow (L^{-1} \otimes M)[-d-e] \xrightarrow{F} M[-e] \longrightarrow M[-e]_{|Y} \longrightarrow 0$$

we deduce the exact sequence

$$(2) \qquad 0 \longrightarrow H^0(Y, M[-e]_{|Y}) \xrightarrow{d^0} H^1(P, (L^{-1} \otimes M)[-d-e]) \xrightarrow{F_1} H^1(P, M[-e])$$

thus $H^0(Y, M[-e]_{|Y}) \cong \ker(F_1)$, where we denote by F_1 the map induced by multiplication by F on the first cohomology groups. Since now

$$H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-d-e)) \qquad (d+e \geq 2)$$

the domain of F_1 is non trivial if and only if $h^0(P_2, L^{-1} \otimes M) \neq 0$. So for suitable choices of line bundles on P_2 we might find interesting, non-complete intersection, codimension two subvarieties of P in this way. In the proof of Proposition 1.4 we explain how to compute F_1.

1.2. **Example.** Let $P_2 = \mathbf{P}^n$, $L = \mathcal{O}_{\mathbf{P}^n}(k)$, $M = \mathcal{O}_{\mathbf{P}^n}(k+l)$ with $l \geq 0$, let $d \geq 1$ and $e = 1$. Then $h^0(\mathbf{P}^n, L^{-1} \otimes M) = h^0(\mathcal{O}_{\mathbf{P}^n}(l)) \neq 0$ so $h^1(P, (L^{-1} \otimes M)[-d-e]) \neq 0$, but $h^1(P, M[-e]) = 0$ since $\mathcal{O}_{\mathbf{P}^1}(-1)$ has no cohomology. Thus $H^0(Y, M[-e]_{|Y}) \cong H^1(P, (L^{-1} \otimes M)[-d-e])$ is indeed non-trivial.

1.3. **Generalized complete intersections.** Given a variety $Y \subset P$ that is the zero locus of $F \in H^0(L[d])$ as in Section 1.1, and given a global section $\tau \in H^0(M[-e]_{|Y})$, its zero locus $X := (\tau) \subset Y$ is called a generalized complete intersection.

The scheme X may not be defined by two global sections σ_1, σ_2 of line bundles L_1, L_2 on P. However in certain cases we can find three sections of line bundles on P which define X:

1.4. **Proposition.** Let $F \in H^0(P, L[d])$, let $Y = (F)$, let $\tau \in H^0(Y, M[-e]_{|Y})$ with $d, e \geq 1$ be as above and assume that $H^1(P_2, L^{-1} \otimes M) = 0$.

Then there are two global sections $G, H \in H^0(P, M[d-1])$ such that the generalized complete intersection subscheme X of P defined by τ in Y can also be defined as

$$X = \{x \in P : F(x) = G(x) = H(x) = 0\}$$

(the equality is of schemes). Moreover, there is a global section $A \in H^0(P, (L^{-1} \otimes M)[d+e-2])$ such that $AF = z_1^{d+e-1}G + z_0^{d+e-1}H$, so that on the open subset of $P_2 \times \mathbf{P}^1$ where $z_0 \neq 0$ the subscheme X of P is defined by the two equations $F = G = 0$.

Proof. We use Cech cohomology to make the isomorphism $H^0(Y, M[-e]_{|Y}) \cong \ker(F_1)$, see exact sequence (2), explicit. Let $U_i \subset \mathbf{P}^1$ be the open subset where $z_i \neq 0$. For a coherent sheaf $\mathcal{G}$ on $\mathbf{P}^1$ we have the exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^1, \mathcal{G}) \longrightarrow \mathcal{G}(U_0) \oplus \mathcal{G}(U_1) \xrightarrow{\delta} \mathcal{G}(U_0 \cap U_1) \longrightarrow H^1(\mathbf{P}^1, \mathcal{G}) \longrightarrow 0 ,$$

where $\delta(t_0, t_1) = t_0 - t_1$. The cohomology groups we consider are computed with the Künneth formula. Note that after tensoring this exact sequence by a vector space W, we obtain that $W \otimes H^0(\mathbf{P}^1, \mathcal{G}) = \ker(1_W \otimes \delta)$ and $W \otimes H^1(\mathbf{P}^1, \mathcal{G}) = \text{coker}(1_W \otimes \delta)$.

For an affine open subset $V \subset \mathbf{P}^1$, the cohomology of the exact sequence (1) on $P_2 \times V$ gives the exact sequence, where we extend $M[-e]_{|Y}$ by zero to $P_2 \times V$,

$$H^0(P_2 \times V, M[-e]) \longrightarrow H^0(P_2 \times V, M[-e]_{|Y}) \longrightarrow H^1(P_2 \times V, (L^{-1} \otimes M)[-d-e]) .$$

The Künneth formula, combined with the assumption $H^1(P_2, L^{-1} \otimes M) = 0$ and the fact that $H^1(V, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ since V is affine, implies that the last group is zero.

Taking $V = U_0, U_1$, the exact sequence (1) on $P_2 \times V$ thus gives two exact sequences whose sum (term by term) is

(3)
$$0 \longrightarrow \oplus_{i=0}^1 H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-d-e)(U_i)) \xrightarrow{F}$$
$$\oplus_{i=0}^1 H^0(M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-e)(U_i)) \longrightarrow \oplus_{i=0}^1 (M[-e]_{|Y})(P_2 \times U_i) \longrightarrow 0 .$$

Similarly taking $V = U_0 \cap U_1$ one has the exact sequence:

(4)
$$0 \longrightarrow H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-d-e)(U_0 \cap U_1)) \xrightarrow{F}$$
$$H^0(M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-e)(U_0 \cap U_1)) \longrightarrow (M[-e]_{|Y})(P_2 \times (U_0 \cap U_1)) \longrightarrow 0 .$$

Next we use the Cech boundary map δ to map sequence (3) to sequence (4) and we obtain a commutative diagram with three complexes as columns. The first two columns are Cech complexes for the covering $\{U_i\}_{i=0,1}$ of $\mathbf{P}^1$, their cohomology groups are respectively

$$H^0(L^{-1} \otimes M) \otimes H^q(\mathcal{O}_{\mathbf{P}^1}(-d-e)) \cong H^q(P, (L^{-1} \otimes M)[-d-e]),$$
$$H^0(M) \otimes H^q(\mathcal{O}_{\mathbf{P}^1}(-e)) \cong H^q(P, M[-e]), \quad (q = 0, 1) .$$

The zero-th cohomology group of the last column is $H^0(Y, M[-e]_{|Y})$. So we conclude that the maps q and F_1 can be computed with the long exact cohomology sequence associated to this diagram.

We observe, but will not use, that the Künneth formula implies that $H^2(P, (L^{-1} \otimes M)[-d - e]) = 0$ and thus the cohomology sequence of (1) gives a six term exact sequence with the zero-th and first cohomology groups. The first 5 terms are the same as those of the long exact sequence associated to the diagram, so we conclude that the first cohomology group of the last column is $H^1(Y, M[-e]_{|Y})$.

Given $\tau \in H^0(Y, M[-e]_{|Y})$, let $q := d^0(\tau) \in \ker(F_1)$. Since the first row (3) of the complex is exact, the section τ is locally given by restricting sections $\tau_i \in M[-e](P_2 \times U_i)$ to Y. By the snake lemma, they satisfy $\tau_0 - \tau_1 = Fq$ on $P_2 \times (U_0 \cap U_1)$, in particular $\tau_0 = \tau_1$ on $Y \cap (P_2 \times (U_0 \cap U_1))$ since $F = 0$ on Y.

The images of the $z_0^{-j} z_1^{-d-e+j} \in \mathcal{O}_{\mathbf{P}^1}(-d - e)(U_0 \cap U_1)$, $j = 1, \ldots, d + e - 1$, form a basis of $H^1(\mathbf{P}^1, \mathcal{O}(-d - e))$. A cohomology class $q \in H^1(P, (L^{-1} \otimes M)[-d - e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbf{P}^1, \mathcal{O}(-d-e))$ can thus be represented by $q = \sum_j q_j z_0^{-j} z_1^{-d-e+j}$ with $q_i \in H^0(P_2, L^{-1} \otimes M)$. Let $F = \sum_i f_i z_0^i z_1^{d-i}$, where $f_i \in H^0(P_2, L)$, then Fq is homogeneous of degree $d - (d + e) = -e$ and it is a sum of terms $r_k z_0^k z_1^{-e-k}$ with $r_k \in H^0(P_2, M)$. Writing

$$Fq = \sum_{k=-d-e+1}^{d-1} r_k z_0^k z_1^{-e-k} = \left(\sum_{k=-d-e+1}^{-e} r_k z_0^k z_1^{-e-k} \right) + \left(\sum_{k=-e+1}^{-1} r_k z_0^k z_1^{-e-k} \right) + \left(\sum_{k=0}^{d-1} r_k z_0^k z_1^{-e-k} \right),$$

the first summand lies in $M[-e](P_2 \times U_0)$ (where $z_0 \neq 0$) and the last summand lies in $M[-e](P_2 \times U_1)$, we denote these summand by τ_0 and $-\tau_1$ respectively. The middle summand has monomials $z_0^a z_1^b$ with both $a, b < 0$. Thus Fq represents a class in $q' \in H^1(P, M[-e])$, which is the same as the class represented by the middle summand. By definition, one has $q' = F_1(q)$ and thus $q \in \ker(F_1)$ when all coefficients r_k, $k = -e + 1, \ldots, -1$, are zero.

Since $q \in \ker(F_1)$ this middle summand is zero, so that $Fq = \tau_0 - \tau_1$ as desired. Now we define $G := z_0^{d+e-1} \tau_0$ and $H := -z_1^{d+e-1} \tau_1$ so that all their monomials $z_0^a z_1^b$ have $a, b \geq 0$ and $a + b = d - 1$, thus both $G, H \in H^0(P, M[d-1])$. Then $(z_0 z_1)^{d+e-1} Fq = z_1^{d+e-1} G + z_0^{d+e-1} H$ and with $A := (z_0 z_1)^{d+e-1} q \in H^0(P, (L^{-1} \otimes M)[d + e - 2])$ we find the desired relation. $\qquad \square$

1.5. Example. With the choices of P_2, L, M as in Example 1.2, and if X is a smooth variety (of dimension $n - 1$), then $H^1(P_2, L^{-1} \otimes M) = H^1(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l)) = 0$, for any l, if $n > 1$. The adjunction formula implies that X has trivial canonical bundle if we choose $l = n + 1 - 2k$ and $d = 3$. In that case $P = \mathbf{P}^n \times \mathbf{P}^1$ and F is homogeneous of bidegree $(k, 3)$ whereas G, H have bidegree $(n + 1 - k, 2)$.

1.6. A fibration on X. Given X as in the proposition, the projection $\pi_2 : P_2 \times \mathbf{P}^1 \to \mathbf{P}^1$ restricts to X to give a fibration denoted by $\pi_2 : X \to \mathbf{P}^1$. For a point $p = (z_0 : z_1) \in \mathbf{P}^1$, we denote by $F_p \in H^0(P_2, L)$, $H_p \in H^0(P_2, M)$ the restrictions of F and H to the fiber X_p. The equation $AF = z_1^{d+e-1} G + z_0^{d+e-1} H$ shows that if $z_1 \neq 0$ then F_p and H_p define the fiber X_p, which is thus a complete intersection in P_2.

1.7. Example. This example illustrates that X, as in Proposition 1.4, might be reducible, even if $h^0(Y, M[-e]_{|Y})$ is rather large. The example is taken from [AAGGL, Table 4], third

item (with $i = 2$) where it is in fact observed that no smooth varieties arise in that case. We take

$$P_2 := \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1 , \qquad L := \mathcal{O}(0,1,1), \quad M := \mathcal{O}(3,1,1) , \qquad d = 4,\ e = -2 .$$

Notice that $H^1(P_2, L^{-1} \otimes M) = H^1(\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(3,0,0)) = 0$ by the Künneth formula, so we can, but will not, apply Proposition 1.4. Since $h^1((L^{-1} \otimes M)[-d-e]) = h^1(\mathcal{O}(3,0,0)[-6]) = 10 \cdot 1 \cdot 1 \cdot 5 = 50$ and $h^1(M[-e]) = 10 \cdot 2 \cdot 2 \cdot 1 = 40$, we find $h^0(M[-e]_{|Y}) \geq 10$. We will show that, for general Y, $h^0(M[-e]_{|Y}) = 10$ but that all sections of $M[-e]_{|Y}$ define reducible subvarieties of Y.

Due to the first zero in $L = \mathcal{O}(0,1,1)$, the variety Y is a product, $Y = \mathbf{P}^2 \times S \subset P$, with $S \subset (\mathbf{P}^1)^3$ the surface defined by a section of $\mathcal{O}(1,1,4)$. Then we have $h^0(M[-e]_{|Y}) = h^0(\mathbf{P}^2 \times S, \pi_1^* \mathcal{O}_{\mathbf{P}^2}(3) \otimes \pi_2^* \mathcal{O}_S(1,1,-2))$ and using the Künneth formula we find $h^0(M[-e]_{|Y}) = h^0(\mathcal{O}_{\mathbf{P}^2}(3)) h^0(\mathcal{O}_S(1,1,-2)) = 10 h^0(\mathcal{O}_S(1,1,-2))$. The exact sequence

$$0 \longrightarrow \mathcal{O}_{(\mathbf{P}^1)^3}(0,0,-6) \xrightarrow{f} \mathcal{O}_{(\mathbf{P}^1)^3}(1,1,-2) \longrightarrow \mathcal{O}_S(1,1,-2) \longrightarrow 0 ,$$

where f is the equation of S, shows that (with f_1 the map induced by f on H^1):

$$h^0(\mathcal{O}_S(1,1,-2)) = \dim \ker \left(f_1 : H^1(\mathcal{O}_{(\mathbf{P}^1)^3}(0,0,-6)) \to H^1(\mathcal{O}_{(\mathbf{P}^1)^3}(1,1,-2)) \right) .$$

Since these spaces have dimensions $1 \cdot 1 \cdot 5 = 5$ and $2 \cdot 2 \cdot 1 = 4$ respectively, one expects $h^0(\mathcal{O}_S(1,1,-2)) = 1$. In that case any section $\tau \in H^0(M[-e]_{|Y})$ would be the product $\tau = gs$ with $g \in H^0(\mathcal{O}_{\mathbf{P}^2}(3))$ and $s \in H^0(\mathcal{O}_S(1,1,-2))$ the unique (up to scalar multiple) section, hence X would be reducible.

To see that indeed $h^0(\mathcal{O}_S(1,1,-2)) = 1$ for a general equation f, take a smooth (genus one) curve C of bidegree $(2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ and choose eight distinct points on C which are not cut out by another curve of bidegree $(2,2)$. As curves of bidegree $(1,4)$ depend on $2 \cdot 5 = 10$ parameters, we can find two polynomials g_0, g_1 of bidegree $(1,4)$ such that $g_0 = g_1 = 0$ consists of these eight points on C. Take $f = x_0 g_0 + x_1 g_1$ with $(x_0 : x_1) \in \mathbf{P}^1$, the first copy of $\mathbf{P}^1$ in $(\mathbf{P}^1)^3$, and the g_i on the last two copies of $\mathbf{P}^1$. The surface $S \subset (\mathbf{P}^1)^3$ defined by f is thus the blow up of $\mathbf{P}^1 \times \mathbf{P}^1$ in the eight points where $g_0 = g_1 = 0$. The adjunction formula shows that the line bundle $\mathcal{O}_S(1,1,-2)$ is the anticanonical bundle of S. The effective anticanonical divisors are the strict transforms of bidegree $(2,2)$-curves on passing through these eight points. Hence the strict transform of C in S will be the unique effective anticanonical divisor on S and therefore $h^0(\mathcal{O}_S(1,1,-2)) = 1$.

2. An example: a generalized complete intersection Calabi-Yau threefold

2.1. We illustrate the use of Proposition 1.4 (and its proof) for the generalized complete intersection Calabi Yau discussed in [AAGGL, Section 2.2.2]. We also consider an explicit example which has a non-trivial involution and we compute the Hodge numbers of a desingularization of the quotient threefold which is again a CY.

2.2. **The varieties P_2 and Y.** We consider the case that $P_2 = \mathbf{P}^4$, we choose the line bundle $L := \mathcal{O}_{\mathbf{P}^4}(2)$ and we let $d = 3$. Then the line bundle $L[d] = \mathcal{O}_P(2,3)$ is very ample on $P = \mathbf{P}^4 \times \mathbf{P}^1$ and thus a general section F will define a smooth fourfold Y of P. To obtain a CY threefold in Y, we consider global sections of the anticanonical bundle of Y. By adjunction, $\omega_Y = (\mathcal{O}_P(-5,-2) \otimes \mathcal{O}_P(2,3))_Y = \mathcal{O}_Y(-3,1)$. Thus we take $M = \mathcal{O}_{\mathbf{P}^4}(3)$ and $e = 1$, so that

$M[-e]_{|Y} = \mathcal{O}_Y(3,-1) = \omega_Y^{-1}$. As the H^1 of any line bundle on $\mathbf{P}^4$ is trivial, we can use (the proof of) Proposition 1.4 to find polynomials $G, H \in H^0(P, \mathcal{O}_P(3,2))$ which together with F define a generalized complete intersection X.

As in Example 1.2, we get

$$H^0(\mathcal{O}_Y(3,-1)) \xrightarrow{\cong} H^1(\mathcal{O}_P(1,-4)) .$$

To find explicit elements of $H^0(\mathcal{O}_Y(3,-1))$, we write the defining equation of Y as

$$F = P_0 z_0^3 + P_1 z_0^2 z_1 + P_2 z_0 z_1^2 + P_3 z_1^3 \qquad (\in H^0(P, \mathcal{O}_P(2,3))) ,$$

with $P_i \in H^0(\mathbf{P}^4, \mathcal{O}(2))$ homogeneous polynomials of degree two in $y = (y_0 : \ldots : y_4)$. As $H^1(\mathcal{O}_P(1,-4)) \cong H^0(\mathcal{O}_{\mathbf{P}^4}(1)) \otimes H^1(\mathcal{O}_{\mathbf{P}^1}(-4))$, a basis of this $5 \cdot 3 = 15$ dimensional vector space are the products of one of $y_0, \ldots, y_4$ with one of $z_0^{-3} z_1^{-1}, z_0^{-2} z_1^{-2}, z_0^{-1} z_1^{-3}$. Thus any class $q \in H^1(\mathcal{O}_P(1,-4))$ has a representative

$$q = Q_0 z_0^{-3} z_1^{-1} + Q_1 z_0^{-2} z_1^{-2} + Q_2 z_0^{-1} z_1^{-3} \qquad (\in H^1(\mathcal{O}_P(1,-4))) ,$$

with linear forms $Q_i \in H^0(\mathbf{P}^4, \mathcal{O}(1))$. As in the proof of Proposition 1.4 we must write:

$$Fq = \tau_0 - \tau_1, \qquad G := z_0^3 \tau_0, \qquad H := -z_1^3 \tau_1 ,$$

with $\tau_i \in \mathcal{O}_P(3,-1)(\mathbf{P}^4 \times U_i)$. So we find

$$
\begin{aligned}
G &= z_0^2(P_1 Q_0 + P_2 Q_1 + P_3 Q_2) + z_0 z_1(P_2 Q_0 + P_3 Q_1) + z_1^2 P_3 Q_0 , \\
H &= -\left(z_0^2 P_0 Q_2 + z_0 z_1(P_0 Q_1 + P_1 Q_2) + z_1^2(P_0 Q_0 + P_1 Q_1 + P_2 Q_2) \right) .
\end{aligned}
$$

2.3. **The base locus of $|-K_Y|$.** In Section 2.2 we showed how to find the global sections of $\omega_Y^{-1} = \mathcal{O}_Y(3,-1)$ explicitly, locally such a section is given by the polynomials G and H. From the formula for F we see that if $x \in \mathbf{P}^4$ and $P_0(x) = \ldots = P_3(x) = 0$, then the curve $\{x\} \times \mathbf{P}^1$ lies in Y. This curve also lies in the zero loci of G and H, for any choice of $Q_0, Q_1, Q_2 \in H^0(\mathcal{O}_{\mathbf{P}^4}(1))$, hence it lies in the base locus of anticanonical system $|-K_Y|$. Since the four quadrics $P_i = 0$ in $\mathbf{P}^4$ intersect in at least 2^4 points, counted with multiplicity, we see that this base locus is non-empty. Thus we cannot use Bertini's theorem to guarantee that there are smooth CY threefolds $X \subset Y$, but we resort to an explicit example, see below.

2.4. **The CY threefold X.** To obtain an explicit example, we choose

$$P_0 := y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad P_1 := y_0^2 + y_4^2, \quad P_2 := y_1^2 + y_3^2, \quad P_3 := y_0^2 + y_1^2 - y_2^2 - y_3^2 - y_4^2,$$

and

$$Q_0 := y_0, \qquad Q_1 := y_1, \qquad Q_2 := y_2 .$$

Using a computer algebra system (we used Magma [M]), one can verify that $Y := (F = 0)$ and $X := (F = G = H = 0)$ are smooth varieties in P. The variety X is a Calabi-Yau threefold since it is an anticanonical divisor on Y. In [AAGGL, (2.27), (2.28)] one finds that the Hodge numbers of X are $(h^{1,1}(X), h^{2,1}(X)) = (2, 46)$, in particular, $h^2(X) = 2$, $h^3(X) = 94$.

2.5. **Parameters.** The CY threefold X is defined by a section $F \in H^0(P, \mathcal{O}_P(2,3))$ and a section $\tau \in H^0(Y, \mathcal{O}_Y(3,-1))$. The first is a vector space of dimension

$$h^0(P, \mathcal{O}_P(2,3)) = h^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2)) \cdot h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3)) = 15 \cdot 4 = 60 ,$$

whereas the second has dimension 15. The group $GL(5, \mathbf{C}) \times GL(2, \mathbf{C})$ acts on $H^0(\mathcal{O}_P(2,3))$ and has dimension $5^2 + 2^2 = 29$. The subgroup of elements $(\lambda I_5, \mu I_2)$ with $\lambda^2 \mu^3 = 1$ acts trivially, so we get $60 - 28 = 32$ parameters for P and next $15 - 1 = 14$ parameters for τ, so we do get $32 + 14 = 46 = h^{2,1}(X)$ parameters for X. So the general deformation of X seems to be again a gCICY of the same type as X. (In [AAGGL], just below (2.28), the dependence of X on $\mathcal{P}$, which gives 32 parameters, seems to have been overlooked.)

2.6. **A CY quotient.** A well-known method to obtain Calabi-Yau threefolds is to consider desingularizations of quotients of such threefolds by finite groups, see for example [CGL]. In the example above, we see that $X \subset \mathbf{P}^4 \times \mathbf{P}^1$ has a subgroup $(\mathbf{Z}/2\mathbf{Z})^2 \subset Aut(X)$ given by the sign changes of y_3 and y_4. We consider the involution

$$\iota : X \longrightarrow X, \qquad \Big((y_0 : \ldots : y_4), (z_0 : z_1) \Big) \longmapsto \Big((y_0 : y_1 : y_2 : -y_3 : -y_4), (z_0 : z_1) \Big) .$$

Its fixed point locus has two components, one defined by $y_3 = y_4 = 0$ and the other by $y_0 = y_1 = y_2 = 0$ in X. The first is a curve in $\mathbf{P}^2 \times \mathbf{P}^1 \subset P$, which is smooth, irreducible and reduced of genus 8 according to Magma. Similarly, the other component is a genus 2 curve in $\mathbf{P}^1_{(y_3:y_4)} \times \mathbf{P}^1_{(z_0:z_1)} \subset P$. In fact, only $F = 0$ provides a non-trivial equation for this curve since $y_0 = y_1 = y_2 = 0$ implies $Q_0 = Q_1 = Q_2 = 0$ and hence $G = H = 0$ on this $\mathbf{P}^1 \times \mathbf{P}^1$. As $F = 0$ defines a smooth curve of bidegree $(2,3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, this curve has genus $(2-1)(3-1) = 2$.

In particular, the singular locus of the quotient X/ι consists of two curves of A_1-singularities. Since the fixed point locus X^ι consists of two curves, we conclude that locally on X the involution is given by $(t_1, t_2, t_3) \mapsto (-t_1, -t_2, t_3)$ in suitable coordinates. Hence ι acts trivially on the nowhere vanishing holomorphic 3-form on the CY threefold X. Thus the blow up Z of X/ι in the singular locus will again be a CY threefold.

We determine the Hodge numbers of Z. To do so, it is more convenient to consider the blow up $\tilde{X}$ of X in the fixed point locus X^ι. The involution extends to an involution $\tilde{\iota}$ on $\tilde{X}$, the fixed point set of $\tilde{\iota}$ consists of the two exceptional divisors and the quotient $\tilde{X}/\tilde{\iota}$ is the same Z. Moreover, $H^i(Z, \mathbf{Q}) \cong H^i(\tilde{X}, \mathbf{Q})^{\tilde{\iota}}$, the $\tilde{\iota}$-invariant subspace.

Standard results on the blow up of smooth varieties in smooth subvarieties (cf. [V, Thm 7.31]) show that $h^2(\tilde{X}) = h^2(X) + 2 = 4$ (due to the two exceptional divisors over the two fixed curves) and $h^3(\tilde{X}) = h^3(X) + 2 \cdot 8 + 2 \cdot 2 = 114$ (the contribution of the H^1 of the fixed curves to H^3 of the blow up). The Lefschetz fixed point formula for $\tilde{\iota}$ gives

$$\chi(\tilde{X}^{\tilde{\iota}}) = \sum_{i=0}^{6} (-1)^i tr(\tilde{\iota}^* | H^i(\tilde{X}, \mathbf{Q})) .$$

Notice that $\tilde{\iota}^*$ is the identity on H^0, H^2, H^4, H^6, in particular $h^2(Z) = \dim H^2(\tilde{X}, \mathbf{Q})^{\tilde{\iota}} = 4$. The fixed points of $\tilde{\iota}$ are the two exceptional divisors, these are $\mathbf{P}^1$-bundles over the exceptional curves hence

$$2(2 - 2 \cdot 2) + 2(2 - 2 \cdot 8) = 1 - 0 + 4 - t_3 + 4 - 0 + 1 \implies t_3 = 42 .$$

If the $+, -$ eigenspaces of $\tilde{\iota}$ on $H^3(\tilde{X}, \mathbf{Q})$ have dimensions a, b respectively, then $a + b = 114$ and $a - b = 42$, thus $a = 78$ and $a = \dim H^3(\tilde{X}, \mathbf{Q})^{\tilde{\iota}} = h^3(Z)$. As Z is a CY threefold it has $h^{3,0}(Z) = 1$ and thus $h^{2,1}(Z) = (78 - 2)/2 = 38$. Other examples of CY threefolds with $(h^{1,1}, h^{2,1}) = (4, 38)$ are already known.

2.7. **A (singular) projective model of Z.** The fibers of $\pi_2 : X \to \mathbf{P}^1$ are K3 surfaces, complete intersections of a quadric and a cubic hypersurface in $\mathbf{P}^4$. The involution ι on X restricts to a Nikulin involution on each smooth fiber. The quotient of such a fiber by the involution will in general be isomorphic to a K3 surface in $\mathbf{P}^2 \times \mathbf{P}^1$, defined by an equation of bidegree $(3, 2)$ (see [GS, Section 3.3]). Using the same method as in that reference, we found that the rational map

$$\mathbf{P}^4 \times \mathbf{P}^1 -- \to \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1, \quad \Big((y_0 : \ldots : y_4), (z_0 : z_1) \Big) \longmapsto \Big((y_0 : y_1 : y_2), (y_3 : y_4), (z_0 : z_1) \Big)$$

factors over X/ι and the image, defined by an equation of multidegree $(3, 2, 2)$, is birational with Z. Using the explicit equation for the image and Magma, we found that the image has 38 singular points.

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