

# Note on curves in a Jacobian

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## 1 Introduction

**1.1** For an abelian variety  $A$  over  $\mathbf{C}$  and a cycle  $\alpha \in CH_d(A)_{\mathbf{Q}}$  we define a subspace  $Z_{\alpha}$  of  $CH_d(A)_{\mathbf{Q}}$  by:

$$Z_{\alpha} := \langle n_* \alpha : n \in \mathbf{Z} \rangle \subset CH_d(J(C))_{\mathbf{Q}}.$$

Results of Beauville imply that  $Z_{\alpha}$  is finite dimensional (cf. 2.5 below). In case  $A = J(C)$ , the jacobian of a curve  $C$ , Ceresa has shown that the cycle  $C - C^- := C - (-1)_* C \in Z_C$  is not algebraically equivalent to zero for generic  $C$  of genus  $g \geq 3$ , which implies that for such a curve  $\dim_{\mathbf{Q}} Z_C \geq 2$ .

In this note we investigate the subspace  $Z_{W_m}$  of  $CH_m(J(C))_{\mathbf{Q}}$ , with  $W_m$  the image of the  $m$ -th symmetric power of  $C$  in  $J(C)$  (so  $W_1 = C$ ). To simplify matters we will actually work modulo algebraic equivalence (rather than linear equivalence, note that translates of a cycle are algebraically equivalent).

**1.2** Let  $Z_{\alpha}/\approx_{alg}$  be the image of  $Z_{\alpha} \subset CH_d(A)_{\mathbf{Q}}$  in  $CH_d(A)_{\mathbf{Q}}/\approx_{alg}$ . A  $d$ -cycle  $\alpha$  is Abel-Jacobi equivalent to zero,  $\alpha \approx_{AJ} 0$ , if  $\alpha$  is homologically equivalent to zero and its image in  $J_d(A)$ , the  $d$ -th primitive Intermediate Jacobian of  $A$ , is zero. Recall that any curve of genus  $g$  is a cover of  $\mathbf{P}^1$  for some  $d \leq \frac{g+3}{2}$ .

**1.3 Theorem.** 1. For any abelian variety  $A$  and any  $\alpha \in CH_d(A)$  we have:

$$\dim_{\mathbf{Q}} (Z_{\alpha}/\approx_{AJ}) \leq 2.$$

2. For any curve of genus  $g$  and  $1 \leq n \leq g - 1$  we have:

$$\dim_{\mathbf{Q}} (Z_{W_{g-n}}/\approx_{alg}) \leq n.$$

3. For a curve  $C$  which is a  $d : 1$ -cover of  $\mathbf{P}^1$  we have:

$$\dim_{\mathbf{Q}} (Z_C / \approx_{alg}) \leq d - 1.$$

(We prove 1.3.1 in 2.5, 1.3.2 in 2.7 and 1.3.3 in 3.6.)

**1.4** Recall that Ceresa showed that the image of  $W_m - W_m^-$  in  $J_m(J(C))$  is non-zero for generic  $C$  of genus  $g \geq 3$  and  $1 \leq m \leq g - 2$ . Therefore 1.3.1 and 1.3.2 for  $n = 2$  are sharp. In case  $C$  is hyperelliptic, so  $C$  is a 2:1 cover of  $\mathbf{P}^1$ , the cycles  $W_m$  and  $W_m^-$  are however algebraically equivalent. Therefore 1.3.3 is sharp for hyperelliptic curves ( $d = 2$ ) and generic trigonal curves ( $d = 3$ ). In case  $C$  is not hyperelliptic nor trigonal (like the generic curve of genus  $g \geq 5$ ), it would be interesting to know if 1.3.3 is actually sharp.

Note that 1.3.1 implies that one cannot use the Intermediate Jacobian anymore to derive new algebraic relations among the cycles in  $Z_C / \approx_{alg}$ . Recently M.Nori [N] constructed cycles on complete intersections in  $\mathbf{P}^N$  which are Abel-Jacobi equivalent to zero but not algebraically equivalent to zero. There is thus the possibility that similar cycles can be found on the Jacobian of a curve of genus  $g \geq 5$ . A cycle in  $Z_C$ , for certain modular curves  $C$ , was investigated by B.H.Gross and C.Schoen, [GS], esp. section 5, see also 2.8.

**1.5** The inequalities 1.3.1 and 1.3.2 are consequences of work of Beauville. The main part of the paper deals with the proof of 1.3.3. Recall that on a smooth surface  $S$  homological and algebraical equivalence for curves coincide. Thus if we have map  $\Phi : S \rightarrow J(C)$  and relation  $a_1[C_1] + \dots + a_n[C_n] = 0$  in  $H^2(S, \mathbf{Q})$ , we get  $a_1\Phi_*C_1 + \dots + a_n\Phi_*C_n \approx_{alg} 0$  in  $J(C)$ . We use this remark to obtain our result, the main difficulty is of course to find suitable surfaces, curves in them and maps to  $J(C)$  and to determine Neron-Severi groups ( $= Im(CH_1(S) \rightarrow H^2(S, \mathbf{Q}))$ ) of the surfaces involved.

**1.6** We are indebted to S.J.Edixhoven and C.Schoen for several helpful discussions.

## 2 General results

**2.1** The effect of  $n_*$  and  $n^*$  on the Chow groups has been investigated by Beauville ([B1], [B2]), Dehninger and Murre [DM] and Künnemann [K] using the Fourier transform on abelian varieties. Below we summarize some of their results and derive the finite dimensionality of  $Z_\alpha$  as well as Thm 1.3.1 and 1.3.2.

**2.2** Let  $A$  be a  $g$ -dimensional abelian variety, we will view  $B := A \times A$  as an abelian scheme over  $A$  using the projection on the first factor:

$$\begin{array}{ccc} B & & A \times A \\ \downarrow & = & \downarrow \pi_1 \\ A & & A \end{array}$$

For each integer  $n$ , we have an  $s_n \in B(A)$  and a cycle  $\Gamma_n \in CH^g(B)$ :

$$s_n : A \longrightarrow B, \quad a \longmapsto (a, na), \quad \Gamma_n := s_{n*}A \in CH^g(A \times A),$$

and  $\Gamma_n$  is, as the notation suggests, the graph of multiplication by  $n$  on  $A$ . The cycle  $\Gamma_n$  defines for each  $i$  a map on  $CH^i(A)$  which is just  $n_*$ :

$$n_* = \Gamma_n : CH^i(A) \longrightarrow CH^i(A), \quad \alpha \longmapsto n_*\alpha = \pi_{2*}(\pi_1^*\alpha \cdot \Gamma_n).$$

Next we introduce a relative version of the Pontryagin product  $*$  on the Chow groups of the  $A$ -scheme  $B$  (cf. [K], (1.2)). Let  $m_B : B \times_A B \rightarrow B$  be the multiplication map, then:

$$\alpha * \beta := m_{B*}(\alpha \times_A \beta), \quad \alpha, \beta \in CH^*(B).$$

Since the relative dimension of  $\Gamma_n$  over  $A$  is 0, the cycle  $\Gamma_n * \Gamma_m$  lies in  $CH^g(B)$  and one has ([K], (1.3.4)):

$$\Gamma_n * \Gamma_m = \Gamma_{n+m} \in CH^g(A \times A)_{\mathbf{Q}}, \quad \text{so} \quad \Gamma_n = \Gamma_1^{*n},$$

where we write  $\alpha^{*n}$  for the  $n$ -fold Pontryagin product of a cycle  $\alpha$  with  $n > 0$  and we put  $\alpha^{*0} := \Gamma_0$ . In [K] 1.4.1 a generalization of a theorem of Bloch is proved, which implies:

$$(\Gamma_1 - \Gamma_0)^{(2g+1)} = 0 \quad (\in CH^g(B)_{\mathbf{Q}}). \quad (2.2.1)$$

Using the ring structure on  $CH^g(B)_{\mathbf{Q}}$  with product  $*$  one can thus define the following cycles  $\pi_i$ ,  $0 \leq i \leq 2g$  in  $CH^g(B)_{\mathbf{Q}}$ :

$$\pi_i := \frac{1}{(2g-i)!} (\log \Gamma_1)^{*(2g-i)},$$

with:

$$\log \Gamma_1 := (\Gamma_1 - \Gamma_0) - \frac{1}{2}(\Gamma_1 - \Gamma_0)^{*2} + \dots + \frac{1}{2g}(\Gamma_1 - \Gamma_0)^{*(2g)}.$$

Let  $\Delta = \Gamma_1 \in CH^g(A \times A)_{\mathbf{Q}}$  the class of the diagonal, then ([K], proof of 3.1.1):

$$\Delta = \pi_0 + \pi_1 + \dots + \pi_{2g}, \quad \pi_i \pi_j = \pi_j \pi_i = \begin{cases} \pi_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \quad (2.2.2)$$

here  $\alpha\beta$ , for cycles  $\alpha, \beta \in CH^g(A \times A)_{\mathbf{Q}}$ , is their product as correspondences:  $\alpha\beta := p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta)$  with the  $p_{ij} : A^3 \rightarrow A^2$  the projection to the  $i, j$  factor. Moreover:

$$\Gamma_n \pi_{2g-i} = \pi_{2g-i} \Gamma_n = n^i \pi_{2g-i}, \quad (2.2.3)$$

(one has  ${}^t\Gamma_n \pi_i = \pi_i {}^t\Gamma_n = n^i \pi_i$  and  ${}^t\pi_i = \pi_{2g-i}$  ([K], 3.1.1), now take transposes).

**2.3 Remark.** We sketch how these results can be obtained from 2.2.1. Let  $M \subset CH^g(B)_{\mathbf{Q}}$  be the subspace spanned by the  $\Gamma_n$ ,  $n \in \mathbf{Z}$ . Then 2.2.1 implies that  $\dim_{\mathbf{Q}} M \leq 2g + 1$  (use  $\Gamma_i * (\Gamma_1 - \Gamma_0)^{*(2g+1)} = 0$  for all  $i \in \mathbf{Z}$ ). Using the Künneth formula, Poincaré duality, one finds that the cohomology class of  $\Gamma_n$  in

$$H^{2g}(A \times A, \mathbf{Q}) = \oplus H^{2g-i}(A, \mathbf{Q}) \otimes H^i(A, \mathbf{Q}) = \oplus \text{Hom}(H^i(A, \mathbf{Q}), H^i(A, \mathbf{Q}))$$

is given by (note  $\Gamma_n$  induces  $n_*$ ):

$$[\Gamma_n] = (n^{2g}, n^{2g-1}, \dots, n, 1) \in \oplus_{i=0}^{2g} \text{Hom}(H^i(A, \mathbf{Q}), H^i(A, \mathbf{Q}))$$

(where  $n^{2g-i}$  in the  $i$ -th component means multiplication by  $n^{2g-i}$  on  $H^i(A, \mathbf{Q})$ ). Therefore  $\dim_{\mathbf{Q}} M / \sim_{\text{hom}} = 2g + 1$  and  $\dim_{\mathbf{Q}} M = 2g + 1$ . Thus we have the (surprising) result that homological and linear equivalence coincide in  $M$ . We can now define  $\pi_i \in M$  by

$$[\pi_i] := (0, \dots, 0, 1, 0, \dots, 0) \in \oplus_i \text{Hom}(H^i(A, \mathbf{Q}), H^i(A, \mathbf{Q}))$$

(1 in  $i$ -th spot) then 2.2.2 and 2.2.3 follow. To express  $\pi_i$  as combination of  $\Gamma_0, \dots, \Gamma_{2g}$ , note that the ( $\mathbf{Q}$ -linear) ring homomorphism:

$$\mathbf{Q}[X] \longrightarrow M \subset CH^g(B)_{\mathbf{Q}}, \quad X^i \longmapsto \Gamma_i = \Gamma_1^{*i}$$

(with  $*$  product on  $CH^g(B)_{\mathbf{Q}}$ ) gives an isomorphism  $\mathbf{Q}[X]/(X-1)^{2g+1} \cong M$ . Since  $\Gamma_n \Gamma_m = \Gamma_{nm}$  (product as correspondences),  $\pi_{2g-i}$  corresponds to a polynomial  $f_{2g-i}$  with:

$$f_{2g-i}(X^n) = n^i f_{2g-i}(X) \quad \text{so} \quad f_{2g-i}(X) := c_i (\log X)^i \bmod (X-1)^{2g+1}$$

and  $c_i \in \mathbf{Q}$  can be determined with a little more work.

**2.4** Since  $\Delta = \Gamma_1 : CH_d(A) \rightarrow CH_d(A)$  is the identity, we get:

$$CH_d(A)_{\mathbf{Q}} = \oplus CH_d(A)_{(i)}, \quad \text{with} \quad CH_d(A)_{(i)} := \pi_{2g-i} CH_d(A)_{\mathbf{Q}}.$$

and each  $CH_d(A)_{(i)}$  is an eigenspace for the multiplication operators:

$$n_* \alpha = n^i \alpha \quad \forall n \in \mathbf{Z}, \quad \forall \alpha \in CH_d(A)_{(i)},$$

a result which was first obtained by Beauville [B2]. Moreover, he proves:

$$CH_d(A)_{(i)} \neq 0 \quad \implies \quad d \leq i \leq d + g \quad (2.4.1)$$

(and gives sharper bounds for some  $d$  in prop.3 of [B2], it has been conjectured that  $CH_d(A)_{(i)} \neq 0 \Leftrightarrow 2d \leq i \leq d + g$ ).

**2.5 Proposition.** Let  $A$  be an abelian variety and let  $\alpha \in CH_d(A)_{\mathbf{Q}}$ . Then:

$$\dim_{\mathbf{Q}} Z_{\alpha} \leq g + 1 \quad \text{and} \quad \dim_{\mathbf{Q}} (Z_{\alpha} / \approx_{AJ}) \leq 2.$$

Moreover, we have  $\pi_i Z_{\alpha} \subset Z_{\alpha}$  for all  $i$ .

**Proof.** We write  $\alpha$  as a sum of weight vectors ( $\alpha_{(i)} := \pi_{2g-i} \alpha$ ):

$$\alpha = \alpha_{(d)} + \alpha_{(d+1)} + \dots + \alpha_{(g+d)}, \quad \text{so} \quad n_* \alpha_{(i)} = n^i \alpha.$$

Taking  $g$  distinct, non-zero, integers  $n_j$ , the determinant of the matrix expressing the  $n_{i*} \alpha \in Z_{\alpha}$  in terms of the  $\alpha_i$  is a Vandermonde determinant. Thus each  $\alpha_{(i)} = \pi_{2g-i} \alpha \in Z_{\alpha}$  and  $Z_{\alpha}$  is spanned by the  $\alpha_{(i)}$ .

Since  $n_*$  acts as  $n^{2d}$  on  $H^{2g-2d}(A, \mathbf{Q})$  and as  $n^{2d+1}$  on  $J_d(A)$ , the space  $Z_{\alpha} / \approx_{AJ}$  is spanned by  $\alpha_{(2d)}$  and  $\alpha_{(2d+1)}$ . □

**2.6** In this section, we fix an abel-jacobi map  $C \hookrightarrow J(C)$ . Note that  $W_d = \frac{1}{d!} C^{*d}$  (Pontryagin product on  $J(C)$ ). Let  $\Theta \in CH^1(J(C))$  be a symmetric theta divisor, so  $\Theta$  is a translate of  $W_{g-1}$ . Let  $\Theta^d \in CH_{g-d}(J(C))$  be the  $d$ -fold intersection of  $\Theta$ .

**2.7 Proposition.** For any  $d$ ,  $1 \leq d \leq g - 1$ , we have  $\Theta^d \in Z_{W_{g-d}}$ , more precisely:

$$\pi_{2g-i} W_{g-d} \neq 0 \implies 2(g-d) \leq i \leq 2g-d, \quad \text{and} \quad \pi_{2d} W_{g-d} = \frac{1}{d!} \Theta^d.$$

Moreover we have:

$$\dim_{\mathbf{Q}} (Z_{W_{g-d}} / \approx_{alg}) \leq d.$$

**Proof.** We first prove the case  $d = g - 1$ . Since the map  $CH_1(C)_{\mathbf{Q}} \rightarrow H^{2g-2}(J(C), \mathbf{Q})$  factors over  $\pi_{2g-2}CH_1(J(C))_{\mathbf{Q}}$  (cf. the proof of 2.5 or even better, [B2], p.650) we know that  $\pi_{2g-2}C \neq 0$ . Then its Fourier transform  $F_{CH}(\pi_{2g-2}C) \in \pi_2CH_{g-1}(J(C))_{\mathbf{Q}} (\cong NS(J(C))_{\mathbf{Q}})$  (cf. [B2], prop.1) is non-zero. For a generic Jacobian  $NS_{\mathbf{Q}}$  is one dimensional and thus  $\pi_2CH_{g-1}(J(C))_{\mathbf{Q}} = \mathbf{Q}\Theta$  ([B1], prop.5 and [B2], prop.1). By specializing, we have for all curves that  $F_{CH}(\pi_{2g-2}C) \in \mathbf{Q}\Theta$ .

From [B1], prop.5 we have  $\Theta = -F_{CH}(\frac{1}{(g-1)!}\Theta^{g-1})$ , so for a nonzero constant  $c$ :

$$F_{CH}(\pi_{2g-2}C) = cF_{CH}(\frac{1}{(g-1)!}\Theta^{g-1})$$

and, by using  $F_{CH}^2 = (-1)^g\Gamma_{-1}$  and comparing cohomology classes, we get:

$$\pi_{2g-2}C = \frac{1}{(g-1)!}\Theta^{g-1}.$$

Next we recall that  $\pi_{2g-1}CH_1(A) = 0$  for any abelian variety  $A$  ([B2], prop.3), thus:

$$C = C_{(2)} + \dots + C_{g+1} \quad \text{with } C_{(i)} = \pi_{2g-i}C,$$

so  $n_*C_{(i)} = n^iC_{(i)}$  and  $C_{(2)} = \frac{1}{(g-1)!}\Theta^{g-1}$ . Therefore

$$W_{g-d} = \frac{1}{(g-d)!}C^{*(g-d)} = \frac{1}{(g-d)!}C_{(2)}^{*(g-d)} + Y \quad (2.7.1)$$

with  $Y$  a sum of cycles  $C_{(i_1)} * \dots * C_{(i_{g-d})}$  with all  $i_j \geq 2$  and at least one  $> 2$ . Therefore, using  $n_*(U * V) = (n_*U) * (n_*V)$ , we get:  $\pi_{2d}W_{g-d} = \frac{1}{(g-d)!}C_{(2)}^{*(g-d)}$  and  $\pi_jW_{g-d} = 0$  for  $j > 2d$ .

By [B1], corr.2 of prop.5,  $\Theta^d$  lies in the subspace spanned by  $(\Theta^{g-1})^{*(g-d)}$ , so  $C_{(2)}^{*(g-d)} = c\Theta^d$  for a non-zero  $c \in \mathbf{Q}$  and taking cohomology classes one finds  $c = 1$ .

Finally, using 2.4.1 and 2.7.1 we can write

$$W_{g-d} = \alpha_{(2g-2d)} + \dots + \alpha_{(2g-d)}, \quad \text{with } \alpha_{(i)} := \pi_{2g-i}W_{g-d},$$

and since  $\alpha_{(2g-d)} \approx_{alg} 0$  by [B2], prop.4a, we get  $\dim_{\mathbf{Q}}(Z_{W_{g-d}} / \approx_{alg}) \leq d$ .  $\square$

**2.8** In the paper [GS] the following cycle (modulo  $\approx_{alg}$ ) is considered:

$$Z := 3_*C - 3 \cdot 2_*C + 3C = (\Gamma_1 - \Gamma_0)^{*3}C.$$

This cycle is related to Ceresa's cycle in the following way:

**2.9 Proposition.** We have:

$$\pi_{2g-2}(Z) = \pi_{2g-2}(C - C^-) = 0 \quad \text{and} \quad \pi_{2g-3}(Z) = 3\pi_{2g-3}(C - C^-).$$

In particular,  $Z$  is abel-jacobi equivalent to  $3(C - C^-)$ .

**Proof.** As we saw before (proof of 2.7) we can write:

$$C = C_{(2)} + C_{(3)} + \dots \quad \text{with } C_{(i)} := \pi_{2g-i}C.$$

Since  $n_*C_{(i)} = n^i C_{(i)}$  we have:

$$C - C^- = 2(C_{(3)} + 0 + C_{(5)} + \dots), \quad Z = 6C_{(3)} + 36C_{(4)} + \dots$$

Therefore  $\pi_{2g-2}(C - C^-) = 3\pi_{2g-2}(Z) = 0$ , so both cycles are homologically equivalent to zero, and  $3\pi_{2g-3}(C - C^-) = \pi_{2g-3}(Z) = 6C_{(3)}$ . Since the abel-jacobi map factors over  $\pi_{2g-3}CH_1(J(C))$  we have that  $Z \approx_{AJ} 3(C - C^-)$ .  $\square$

### 3 Proof of 1.3.3

**3.1** Let  $C$  be a generic  $d:1$  cover of  $\mathbf{P}^1$ , and denote by  $g_d^1$  the corresponding linear series. For an integer  $n$ ,  $0 < n < d$  we define a curve  $G_n$  in the  $n$ -fold  $C^{(n)} = \text{Sym}^n(C)$ :

$$G_n = G_n(g_d^1) = \{x_1 + x_2 + \dots + x_n \in C^{(n)} : x_1 + x_2 + \dots + x_n < g_d^1\},$$

see [ACGH], p.342 for the definition of the scheme-structure on  $G_n$ . We define a surface by:

$$S_n = S_n(g_d^1) = G_n \times C.$$

**3.2 Proposition.** Let  $C$  be a generic  $d:1$ -covering of  $\mathbf{P}^1$  of genus  $g \geq 1$ . Then  $S_n$  is a smooth, irreducible surface and

$$\dim_{\mathbf{Q}} NS(S_n)_{\mathbf{Q}} = 3.$$

**Proof.** Since  $C$  is generic, the curve  $G_n$  is irreducible (consider the monodromy representation) and for a simple covering the smoothness of  $G_n$  follows from a local computation. Therefore also  $S_n$  is smooth and irreducible.

Note that for a product of 2 curves  $S_n = G_n \times C$  we have that

$$NS(S_n)_{\mathbf{Q}} = \mathbf{Q}C \oplus \mathbf{Q}G_n \oplus \text{Hom}(J(C), J(G_n))_{\mathbf{Q}}.$$

Using Hodge structures, the last part are just the Hodge-cycles in  $H^1(C, \mathbf{Q}) \otimes H^1(G_n, \mathbf{Q})$ . Since  $J(C)$  and  $Pic^0(C^{(n)})$  are isogeneous (their  $H^1(\mathbf{Q})$ 's are

isomorphic Hodge structures), composing such an isogeny with the pull-back  $Pic^0(C^{(n)}) \rightarrow J(G_n)$ , we have a non-trivial map  $J(C) \rightarrow J(G_n)$ . To show that  $\dim_{\mathbf{Q}} \text{Hom}(J(C), J(G_n))_{\mathbf{Q}} \leq 1$  we use a degeneration argument. First we recall that the Jacobian of  $C$  is simple and has in fact  $\text{End}(J(C)) = \mathbf{Z}$ .

In case  $g = 2$  this is clear since any genus 2 curve can be obtained by deformation from a given  $d:1$  cover and  $C$  is generic. In case  $g = 3$  one reasons similarly, taking the necessary care for the hyperelliptic curves. In case  $g(C) = g > 3$ , assume first that there is an elliptic curve in  $J(C)$ . Specializing  $C$  to a reducible curve with two components  $C'$  and  $C''$ , both of genus  $\geq 2$ , and themselves generic  $d:1$ -covers we obtain a contradiction by induction. Assume now that there is no elliptic curve in  $J(C)$ , then we specialize  $C$  to  $E \times C'$ , with  $E$  an elliptic curve and  $C'$  a generic  $d:1$ -covering of genus  $g - 1$ . By induction again,  $J(C')$  is simple, which contradicts the existence of abelian subvarieties in  $J(C)$ . We conclude that  $J(C)$  is simple. Thus  $\text{End}_{\mathbf{Q}}(J(C))$  is a division ring and specializing again we find it must be  $\mathbf{Q}$ .

We may assume that the  $g_d^1$  exhibits  $C$  as a simple cover of  $\mathbf{P}^1$ , then  $G_n$  is a cover of  $\mathbf{P}^1$  (of degree  $\binom{d}{n}$ ) with only twofold ramification points. Letting two branch points coincide, we obtain a curve  $\overline{C}$  with a node, the normalization  $C'$  of  $\overline{C}$  has genus  $g - 1$  and is again exhibited as a generic  $d : 1$  cover of  $\mathbf{P}^1$ . The curve  $G_n$  acquires  $\binom{d-2}{n-1}$  nodes (since twice that number of branch points coincide pairwise) and the normalization of that curve,  $\overline{G}_n$ , is  $G'_n$ , the curve obtained from the  $g_d^1$  on  $C'$ . The (generalized) Jacobians of  $\overline{C}$ ,  $\overline{G}_n$  are extensions of the abelian varieties (of  $\dim \geq 1$ )  $J(C')$ ,  $J(G'_n)$  by multiplicative groups.

Since the number of simple factors of  $J(G'_n)$  which are isogeneous to  $J(C')$  is greater than or equal to the number of simple factors of  $J(G_n)$  which are isogeneous to  $J(C)$ , and  $J(C)$ ,  $J(C')$  have  $\text{End}_{\mathbf{Q}} = \mathbf{Q}$  it follows:

$$\dim_{\mathbf{Q}} \text{Hom}(J(C), J(G_n))_{\mathbf{Q}} \leq \dim_{\mathbf{Q}} \text{Hom}(J(C'), J(G'_n))_{\mathbf{Q}}.$$

Therefore it suffices to show that for a generic elliptic curve  $C = E$  and a generic  $g_d^1$  on  $E$  we have  $\dim_{\mathbf{Q}} \text{Hom}(E, J(E_n))_{\mathbf{Q}} \leq 1$  (with  $E_n = G_n(g_d^1)$  and  $C = E$ ).

We argue again by induction, using degeneration. Note that for any  $C$ ,  $G_1(g_d^1) \cong G_{d-1}(g_d^1) \cong C$ . Since a generic elliptic curve has  $\text{Hom}(E, E) = \mathbf{Z}$  the statement is true for  $d \leq 3$  (and any  $0 < n < d$ ) and since  $E_1 \cong E$  it is also true for  $n = 1$  (and any  $d \geq 2$ ).

We fix  $E$ , a generic elliptic curve, but let the  $g_d^1$  acquire a base point  $y \in E$ . Then  $E_n(g_d^1)$ ,  $n \geq 2$ , becomes a reducible curve  $\bar{E}_n$ , having two components  $E'_n = E_n(g_{d-1}^1)$  and  $E'_{n-1} = E_{n-1}(g_{d-1}^1)$  which meet transversally

in  $N = \binom{d-2}{n-1}$  points. Indeed, let the divisor of the  $g_{d-1}^1$  containing  $y$  be  $y + y_1 + \dots + y_{d-2}$ , then the points  $y_{i_1} + \dots + y_{i_{n-1}} \in E'_{n-1}$  and  $y + y_{i_1} + \dots + y_{i_{n-1}} \in E'_n$  are identified.

Thus  $J(\overline{E}_n)$  is an extension of  $J(E'_n) \times J(E'_{n-1})$  by the multiplicative group  $(\mathbf{C}^*)^{N-1}$ , in fact there is an exact sequence:

$$1 \longrightarrow \mathbf{C}^* \xrightarrow{\Delta} (\mathbf{C}^*)^N \longrightarrow J(\overline{E}_n) \xrightarrow{\pi} J(E'_n) \times J(E'_{n-1}) \longrightarrow 0,$$

where  $\Delta$  is the diagonal embedding. We will write  $\pi = (\pi_0, \pi_1)$ .

We will prove that

$$\mathrm{Hom}(E, J(\overline{E}_n))_{\mathbf{Q}} \longrightarrow \mathrm{Hom}(E, J(E'_n))_{\mathbf{Q}}, \quad \tilde{\phi} \mapsto \pi_0 \circ \tilde{\phi}$$

is injective. By induction we may assume that  $\dim \mathrm{Hom}(E, J(E'_n))_{\mathbf{Q}} \leq 1$ , thus also  $\dim_{\mathbf{Q}} \mathrm{Hom}(E, J(\overline{E}_n))_{\mathbf{Q}} \leq 1$  and since  $\dim_{\mathbf{Q}} \mathrm{Hom}(E, J(E'_n))_{\mathbf{Q}} \leq \dim_{\mathbf{Q}} \mathrm{Hom}(E, J(\overline{E}_n))_{\mathbf{Q}}$  the assertion on the rank of the Neron-Severi group follows.

Assume that  $\tilde{\phi} \neq 0$ , but  $\pi_0 \circ \tilde{\phi} = 0$ . Then  $\tilde{\phi}(E) \subset \bar{J} := \pi_1^{-1}(J(E'_{n-1}))$ . Pulling back this  $(\mathbf{C}^*)^{N-1}$ -bundle  $\bar{J}$  over  $J(E'_{n-1})$  to  $E$  along  $\pi_1 \circ \tilde{\phi}$ , we obtain a  $(\mathbf{C}^*)^{N-1}$ -bundle  $\tilde{E}$  over  $E$ . The map  $\tilde{\phi} : E \rightarrow \bar{J}$  gives a section of  $\tilde{E} \rightarrow E$ , thus  $\tilde{E}$  is a trivial  $(\mathbf{C}^*)^{N-1}$ -bundle. We show that this gives the desired contradiction.

$$\begin{array}{ccccccc} 1 & \rightarrow & (\mathbf{C}^*)^{N-1} & \rightarrow & \bar{J} & \rightarrow & J(E'_{n-1}) \rightarrow 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \pi_1 \tilde{\phi} \\ 1 & \rightarrow & (\mathbf{C}^*)^{N-1} & \rightarrow & \tilde{E} & \rightarrow & E \rightarrow 0 \end{array}$$

For any distinct  $P, Q \in E'_n$  which are in  $E'_n \cap E'_{n-1}$ , the  $(\mathbf{C}^*)^{N-1}$ -bundle  $\bar{J}$  over  $J(E'_{n-1})$  has a quotient  $\mathbf{C}^*$ -bundle  $\bar{J}_{PQ}$  whose extension class is  $P - Q \in \mathrm{Pic}^0(E'_n) = \mathrm{Ext}^1(J(E'_n), \mathbf{C}^*)$ . By induction, there is a ‘unique’ map in  $\mathrm{Hom}(J(E'_{n-1}), E)$  which must thus be induced by  $x_1 + \dots + x_{n-1} \mapsto x_1 + \dots + x_{n-1} - (n-1)p$ ,  $E'_{n-1} \rightarrow E$  for some  $p \in E$ . Taking  $P = y_1 + \dots + y_{n-1}$  and  $Q = y_2 + \dots + y_{n-1} + y_n$ , the pull-back of  $P - Q$  to  $E = \mathrm{Pic}^0(E)$  is  $y_1 - y_n$ . Choosing the degeneration suitably we may assume that  $y_1 - y_n$  is not a torsion point on  $E$  and thus  $\bar{J}_{PQ}$  has a nontrivial pull-back to  $E$ , contradicting the fact that  $\tilde{E}$ , the pull-back of  $\bar{J}$  to  $E$ , is trivial.  $\square$

**3.3** We define a curve  $\Delta_n$  in the surface  $S_n = G_n \times C$ :

$$\Delta_n := \{(x_1 + x_2 + \dots + x_n, p) \in S_n : p \in \{x_1, \dots, x_n\}\}$$

and for  $n + 1 \leq d$ , another curve  $H_n$  in  $S_n$ :

$$H_n = \{(x_1 + x_2 \dots + x_n, p) \in S_n : x_1 + x_2 + \dots + x_n + p < g_d^1\}.$$

Finally we define the map:

$$\Phi_{l,k}^n : S_n \longrightarrow J(C), \quad (3.3.1)$$

$$(x_1 + x_2 \dots + x_n, p) \longmapsto l(x_1 + x_2 + \dots + x_n) + kp - D_{nl+k},$$

where  $D_{nl+k}$  is some divisor of degree  $nl + k$  on  $C$ . Finally we denote by

$$u_n : C^{(n)} \longrightarrow J(C), \quad D \longmapsto D_n,$$

with  $D_n$  some divisor of degree  $n$ , the Abel-Jacobi map on  $C^{(n)}$ . We simply write  $C$  for  $u_{1*}C$ .

**3.4 Proposition.** Let  $C$  be a generic  $d:1$  cover of  $\mathbf{P}^1$  of genus  $g \geq 1$ . Then:

1.

$$NS(S_n)_{\mathbf{Q}} = \langle C, G_n, \Delta_n \rangle.$$

2. In  $NS(S_n)$  we have:

$$H_n = \binom{d}{n}C + dG_n - \Delta_n. \quad (3.4.1)$$

3. The image of  $G_n$  in  $J(C)$  is a combination of  $C, 2_*C, \dots, n_*C$ :

$$u_{n*}G_n \approx_{alg} \binom{d}{n-1}C - \frac{1}{2}\binom{d}{n-2}2_*C + \dots + \frac{(-1)^{n-1}}{n}\binom{d}{0}n_*C. \quad (3.4.2)$$

**Proof.** The first part follows from the previous proposition and from the fact that in the proof of 3.4.1 we will see that  $H_n$  can be uniquely expressed as a combination of  $C, G_n$  and  $\Delta_n$ .

The proof of 2 and 3 is by induction. In the case  $n = 1$ , so  $G_1 \cong C$ , 3.4.2 is trivial and for 3.4.1 we have  $S_1 = C \times C$  and we must show  $H_1 = d(C + G_1) - \Delta$ . Note the following intersection numbers:

$$H_1 \cdot C = H_1 \cdot G_1 = d - 1, \quad H_1 \cdot \Delta = 2(g + d - 1)$$

(use that the  $g_d^1$  exhibits  $C$  as  $d:1$  cover of  $\mathbf{P}^1$  with  $2(g + d - 1)$  simple ramification points), which imply the result.

Suppose that 3.4.1 and 3.4.2 are true for all  $k < n$ . Note that  $u_n(G_n) = \Phi_{1,1}^{n-1}(H_{n-1})$  and that map  $\Phi_{1,1}^{n-1}$  restricted to  $H_{n-1}$  is  $n:1$  so, using 3.4.1 for  $n - 1$ , we have in  $J(C)$ :

$$G_n \approx_{alg} \frac{1}{n} \left( \binom{d}{n-1}C + dG_{n-1} - (\Phi_{1,1}^{n-1})_*\Delta_{n-1} \right). \quad (3.4.3)$$

where we write  $G_k$  for  $u_{k*}G_k$ . Next we observe that

$$(\Phi_{1,l}^k)_*\Delta_k = (\Phi_{1,l+1}^{k-1})_*H_{k-1} \quad \text{and} \quad (\Phi_{1,l+1}^{k-1})_*C = (l+1)_*C. \quad (3.4.4)$$

Using 3.4.1 in the cases  $n-2, n-3, \dots, 1$  we get:

$$\begin{aligned} (\Phi_{1,1}^{n-1})_*\Delta_{n-1} &\approx_{alg} (\Phi_{1,2}^{n-2})_*H_{n-2} \\ &\approx_{alg} \binom{d}{n-2}2_*C + dG_{n-2} - (\Phi_{1,3}^{n-3})_*H_{n-3} \\ &\approx_{alg} \binom{d}{n-2}2_*C - \binom{d}{n-3}3_*C + d(G_{n-2} - G_{n-3}) + (\Phi_{1,4}^{n-4})_*H_{n-4} \\ &\approx_{alg} \binom{d}{n-2}2_*C - \binom{d}{n-3}3_*C + \dots + (-1)^{n-2}\binom{d}{0}n_*C + \\ &\quad + d(G_{n-2} - G_{n-3} \dots + (-1)^{n-2}G_2 + (-1)^{n-1}G_1) \end{aligned} \quad (3.4.5)$$

(note that  $(\Phi_{1,n-1}^1)_*H_1 \approx_{alg} \binom{d}{1}(n-1)_*C + dG_1 - \binom{d}{0}n_*C$ ). Substitute the expression for  $(\Phi_{1,1}^{n-1})_*\Delta_{n-1}$  from 3.4.5 into 3.4.3:

$$\begin{aligned} G_n &\approx_{alg} \\ &\frac{1}{n} \left( \binom{d}{n-1}C - \binom{d}{n-2}2_*C \dots + (-1)^{n-2}\binom{d}{1}(n-1)_*C + (-1)^{n-1}\binom{d}{0}n_*C \right. \\ &\quad \left. + d(G_{n-1} - G_{n-2} + \dots + (-1)^{n-3}G_2 + (-1)^{n-2}G_1) \right). \end{aligned} \quad (3.4.6)$$

Using the formula 3.4.2 for  $G_k$ ,  $k = 1, 2, \dots, n-1$  and the relation  $\sum_{k=0}^l (-1)^k \binom{d}{l-k} = \binom{d-1}{l}$ , (cf. 4.2.1) we get:

$$\begin{aligned} &G_{n-1} - G_{n-2} + \dots + (-1)^{n-2}G_1 \\ &\approx_{alg} \binom{d-1}{n-3}C - \frac{1}{2}\binom{d-1}{n-4}2_*C + \dots + (-1)^{n-3}\frac{1}{n-1}\binom{d-1}{0}(n-1)_*C. \end{aligned} \quad (3.4.7)$$

Substituting this in formula 3.4.6 and using the identity:

$$\frac{1}{n} \left[ \binom{d}{n-k} + \frac{d}{k} \binom{d-1}{n-k+1} \right] = \frac{1}{k} \binom{d}{n-k}$$

we obtain 3.4.2.

To obtain 3.4.1, note that by (1), there are  $a, b, c \in \mathbf{Q}$  such that:

$$H_n = aC + bG_n + c\Delta_n. \quad (3.4.8)$$

To find them we compute the homology classes of the curves in 3.4.8 in  $H_2(J(C), \mathbf{Q})$  and the intersection numbers of  $H_n$  with  $C$  and  $G_n$ .

For the generic curve  $C$ , the homology class  $[B]$  of a curve  $B$  in  $J(C)$  is a multiple of the class  $[C]$  of  $C$  and this multiple is  $\frac{1}{g}\Theta \cdot B$ . We apply this to  $B = G_n$ . Using the map  $u : C^{(n)} \rightarrow J(C)$ , we have  $u_*(G_n) \cdot \Theta = u_*(G_n \cdot \theta)$  with  $\theta$  the pull-back of  $\Theta$  to  $C^{(n)}$ .

The homology class of  $G_n$  in  $H_2(C^{(n)}, \mathbf{Q})$  is:

$$G_n = \sum_{k=0}^{n-1} \binom{d-g-1}{k} \frac{x^k \theta^{n-k-1}}{(n-1-k)!}, \quad \text{so} \quad G_n \cdot \theta = \sum_{k=0}^{n-1} \binom{d-g-1}{k} \frac{x^k \theta^{n-k}}{(n-1-k)!}, \quad (3.4.9)$$

cf. formula (3.2) on pag.342 of [ACGH] (here  $x$  is the class of the divisor  $C^{(n-1)}$  in  $C^{(n)}$ ). From pag.343 of [ACGH] one has:  $u_*(x^{n-i}\theta^i) = \binom{g}{i}i![\Theta^g]/g!$  and since  $[\Theta^g]/g!$  is the positive generator of  $H^{2g}(J(C), \mathbf{Z})$ , we find:

$$\begin{aligned} G_n \cdot \Theta &= \sum_{k=0}^{n-1} \binom{d-g-1}{k} \binom{g}{n-k} \frac{(n-k)!}{(n-1-k)!} \\ &= g \sum_{k=0}^{n-1} \binom{d-g-1}{k} \binom{g-1}{n-k-1} \\ &= g \binom{d-2}{n-1}. \end{aligned} \quad (3.4.10)$$

Therefore:

$$[G_k] = \binom{d-2}{k-1} [C]. \quad (3.4.11)$$

Next we compute the homology class of  $\Delta_n$ . We use 3.4.4 for  $k = n$ ,  $l = 1$ :  $(\Phi_{1,1}^n)_* \Delta_n = (\Phi_{1,2}^{n-1})_* H_{n-1}$ . By induction, similar to 3.4.5 and using 3.4.7 (with  $n-1$  replaced by  $n$ ) we have:

$$\begin{aligned} (\Phi_{1,1}^n)_* \Delta_n &\approx_{alg} d \left( G_{n-1} - G_{n-2} + \dots (-1)^{n-3} G_2 + (-1)^{n-2} G_1 \right) \\ &\quad + \binom{d}{n-1} 2_* C - \binom{d}{n-2} 3_* C + \dots + (-1)^{n-1} \binom{d}{0} (n+1)_* C \\ &\approx_{alg} d \sum_{k=1}^{n-1} (-1)^{k-1} \frac{1}{k} \binom{d-1}{n-1-k} k_* C + \sum_{k=2}^{n+1} (-1)^k \binom{d}{n+1-k} k_* C. \end{aligned} \quad (3.4.12)$$

Taking the homology classes in  $J(C)$  we get:

$$[\Phi_{1,1}^n(\Delta_n)] = \left( d \sum_{k=1}^{n-1} (-1)^{k-1} k \binom{d-1}{n-1-k} + \sum_{k=2}^{n+1} (-1)^k k^2 \binom{d-1}{n+1-k} \right) [C],$$

Using 4.2.3:

$$\sum_{k=2}^{n+1} (-1)^k \binom{d}{n+1-k} k^2 = \binom{d}{n} + \sum_{k=0}^{n+1} (-1)^k \binom{d}{n+1-k} k^2 = \binom{d}{n} + \binom{d-3}{n-1} - \binom{d-3}{n},$$

we find:

$$[\Phi_{1,1}^n(\Delta_n)] = \left( d \binom{d-3}{n-2} + \binom{d}{n} + \binom{d-3}{n-1} - \binom{d-3}{n} \right) [C]$$

and because of

$$\binom{d-3}{n-1} - \binom{d-3}{n} = d \binom{d-3}{n-1} - (n+1) \binom{d-2}{n-1}, \quad \binom{d-2}{n-1} = \binom{d-3}{n-1} + \binom{d-3}{n-2}$$

we finally obtain:

$$[\Phi_{1,1}^n(\Delta_n)] = \left( d \binom{d-2}{n-2} - (n+1) \binom{d-2}{n} + \binom{d}{n} \right) [C]. \quad (3.4.13)$$

Applying  $(\Phi_{1,1}^n)_*$  to 3.4.8 and taking homology classes we get the following equation for  $a, b, c$ :

$$(n+1) \binom{d-2}{n} = a + b \binom{d-2}{n-1} + c \left( d \binom{d-2}{n-1} - (n+1) \binom{d-2}{n} + \binom{d}{n} \right).$$

On the other hand, we have the following intersections numbers in  $S_n$ :

$$C \cdot C = G_n \cdot G_n = 0, \quad C \cdot G_n = 1, \quad C \cdot \Delta_n = n, \quad G_n \cdot \Delta_n = \binom{d-1}{n-1}$$

and

$$H_n \cdot C = d - n, \quad H_n \cdot G_n = \binom{d-1}{n}.$$

Therefore, by intersecting 3.4.8 with  $C$  and  $G_n$  respectively, we find two more equations for  $a, b, c$ :

$$d - n = b + cn, \quad \binom{d-1}{n} = a + c \binom{d-1}{n-1},$$

and 3.4.1 now follows.  $\square$

**3.5** With these results it is easy to find many relations between the cycles  $n_*C$ . It is a little surprising that the ones we find are equivalent to  $\pi_i C \approx_{alg} 0$  for  $i < 2g - d$  and  $\pi_{2g-1} C \approx_{alg} 0$ .

**3.6 Proposition.** Let  $C$  be a  $d:1$  covering of  $\mathbf{P}^1$ . Then:

$$\pi_{2g-i} C \not\approx_{alg} 0 \implies 2 \leq i \leq d.$$

The cycles  $\pi_{2g-2} C, \dots, \pi_{2g-d} C$  span  $Z_C / \approx_{alg}$  and thus

$$\dim_{\mathbf{Q}} (Z_C / \approx_{alg}) \leq d - 1.$$

For any set  $\{n_1, \dots, n_{d-1}\}$  of non-zero distinct integers the cycles  $n_{1*}C, \dots, n_{d-1*}C$  also span  $Z_C / \approx_{alg}$ .

**Proof.** First of all we show that for all  $n \in \mathbf{Z}$  we have:

$$P_n C := (n+1)_* C - \binom{d}{d-1} n_* C + \dots (-1)^d \binom{d}{0} (n-d+1)_* C + F_d \approx_{alg} 0 \quad (3.6.1)$$

with  $P_n = \Gamma_{n+1} - \binom{d}{d-1} \Gamma_n + \dots \in CH^g(A \times A)$  and with a cycle  $F_d$  depending on  $d$  but not on  $n$ :

$$F_d = d[-C + \binom{d-1}{d-3} C^- - \frac{1}{2} \binom{d-1}{d-4} 2_* C^- + \dots (-1)^{d-1} \frac{1}{d-2} \binom{d-1}{0} (d-2)_* C^-].$$

This relation follows from the easily verified identity (for all  $n \in \mathbf{Z}$ ):

$$\left(\Phi_{1,n}^1\right)_* H_1 = \left(\Phi_{-1,n-1}^{d-2}\right)_* H_{d-2}.$$

Indeed, the l.h.s is by 3.4.1:

$$\left(\Phi_{1,n}^1\right)_* H_1 \approx_{alg} \left(\Phi_{1,n}^1\right)_* (dC + dG_1 - \Delta_1) \approx_{alg} dn_* C + dC - (n+1)_* C,$$

while for the r.h.s. we use  $(\Phi_{-1,l}^k)_* \Delta_k = (\Phi_{-1,l-1}^{k-1})_* H_{k-1}$ :

$$\begin{aligned} & \left(\Phi_{-1,n-1}^{d-2}\right)_* H_{d-2} \\ & \approx_{alg} \left(\begin{smallmatrix} d \\ d-2 \end{smallmatrix}\right)(n-1)_* C + dG_{d-2}^- - \left(\Phi_{-1,n-2}^{d-3}\right)_* H_{d-3} \\ & \approx_{alg} \left(\begin{smallmatrix} d \\ d-2 \end{smallmatrix}\right)(n-1)_* C - \left(\begin{smallmatrix} d \\ d-3 \end{smallmatrix}\right)(n-2)_* C + \dots + (-1)^{d-2}(n-d+1)_* C + \\ & \quad + d(G_{d-2}^- - G_{d-3}^- + \dots + (-1)^{d-3}G_1^-) \end{aligned} \tag{3.6.2}$$

(cf. also the proof of 3.4.2). From this 3.6.1 follows by using 3.4.7.

A more convenient set of relations is obtained from 3.6.1 as follows:

$$(P_n - P_{n-1})C \tag{3.6.3}$$

$$= (n+1)_* C - (d+1)n_* C + \left(\begin{smallmatrix} d+1 \\ d-1 \end{smallmatrix}\right)(n-1)_* C + \dots + (-1)^{d+1}(n-d)_* C$$

$$= \left(\Gamma_{m+d+1} - \left(\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right)\Gamma_{m+d} + \left(\begin{smallmatrix} d+1 \\ 2 \end{smallmatrix}\right)\Gamma_{m+d-1} \dots + (-1)^{d+1}\Gamma_m\right) C \tag{3.6.4}$$

$$\approx_{alg} 0,$$

where we substituted:  $n := m + d$ . Relation 3.6.4 can be rewritten as:

$$\Gamma_m * (\Gamma_1 - \Gamma_0)^{*(d+1)} C \approx_{alg} 0 \quad \forall m \in \mathbf{Z}. \tag{3.6.5}$$

Next we look at the expansion of the  $\pi_i$ 's in  $\Gamma_1 - \Gamma_0$ :

$$\pi_i = \frac{1}{(2g-i)!} (\log \Gamma_1)^{*(2g-i)} =$$

$$= (\Gamma_1 - \Gamma_0)^{*(2g-i)} * \left(a_{2g-i}(\Gamma_1 - \Gamma_0)^{*0} + \dots + a_{2g}(\Gamma_1 - \Gamma_0)^{*i}\right).$$

Therefore  $\pi_i C \approx_{alg} 0$  if  $2g - i \geq d + 1$  so  $\pi_{2g-i} C \approx_{alg} 0$  if  $i \geq d + 1$ . Since  $\pi_{2g} = \Gamma_0$  we also have  $\pi_{2g} C = 0$ , and from [B2], prop.3 we know  $C_{(1)} = 0$  thus:

$$C \approx_{alg} C_{(2)} + C_{(3)} + \dots + C_{(d)} \quad \text{with} \quad C_{(i)} := \pi_{2g-i} C$$

and we conclude that  $\dim_{\mathbf{Q}}(Z_C / \approx_{alg}) \leq d - 1$ . We can in fact obtain  $C_{(1)} \approx_{alg} 0$  from 3.6.1, because a somewhat tedious computation shows that, for some  $c \in \mathbf{Q}$ :

$$\Gamma_{-1}(\frac{1}{d-1}P_0 - \frac{1}{d}P_{-1}) = c\Gamma_0 + \log \Gamma_1 \quad \text{mod } (\Gamma_1 - \Gamma_0)^{*(d+1)} * CH^g(A \times A),$$

here  $\Gamma_{-1}\Gamma_n = \Gamma_{-n}$  is the product as correspondences, the equality is in (a quotient of) the ring  $CH^g(A \times A)$  with  $*$ -product. Since  $\Gamma_0$  acts trivially on one cycles and  $P_n C \approx_{alg} 0$  the statement follows. In fact one shows that both sides are equal to, for some  $c_1 \in \mathbf{Q}$ :

$$\sum_{n=1}^d (-1)^{n-1} \frac{1}{n} \binom{d}{n} \Gamma_n + c_1 \Gamma_0.$$

(Comparing this expression with the one for  $u_{d*}G_d$  in 3.4.2 we see that  $\pi_{2g-1}C \approx_{alg} u_{d*}G_d \approx_{alg} 0$  since  $G_d \cong \mathbf{P}^1$  maps to a point in  $J(C)$ , note that some care must be taken as we defined  $G_n$  only for  $n < d$ ).

Finally we observe that since  $n_*C_{(i)} = n^i C_{(i)}$ , the determinant of the matrix expressing the  $n_{i*}C$  in the  $C_{(i)}$  is a Vandermonde determinant, which is nonzero.  $\square$

**3.7 Proposition.** 1. For a curve  $C$  with  $\pi_i C \approx_{alg} 0$  for  $i \leq 2g - 4$  (for example any curve with a  $g_3^1$ ) we have, with  $C_{(i)} := \pi_{2g-i}C$ :

$$\begin{aligned} C_{(2)} &\approx_{alg} \frac{1}{2}(C + C^-) \\ C_{(3)} &\approx_{alg} \frac{1}{2}(C - C^-) \end{aligned}$$

$$\text{Furthermore: } n_*C \approx_{alg} \frac{n^3+n^2}{2}C - \frac{n^3-n^2}{2}C^-.$$

2. For a curve  $C$  with  $\pi_i C \approx_{alg} 0$  for  $i \leq 2g - 5$  (for example any curve with a  $g_4^1$ ) we have:

$$\begin{aligned} C_{(2)} &\approx_{alg} \frac{-1}{12}(2_*C - 12C - 4C^-) \\ C_{(3)} &\approx_{alg} \frac{1}{2}(C - C^-) \\ C_{(4)} &\approx_{alg} \frac{1}{12}(2_*C - 6C + 2C^-). \end{aligned}$$

**Proof.** We give the proof of the second statement, the first being similar but easier. By assumption we have  $(C_{(i)} := \pi_{2g-i}C)$ :

$$C \approx_{alg} C_{(2)} + C_{(3)} + C_{(4)} \quad \text{with} \quad n_*C_{(i)} = n^i C_{(i)}.$$

Therefore:

$$\begin{aligned} C &\approx_{alg} C_{(2)} + C_{(3)} + C_{(4)} \\ C^- &\approx_{alg} C_{(2)} - C_{(3)} + C_{(4)} \\ 2_*C &\approx_{alg} 4C_{(2)} + 8C_{(3)} + 16C_{(4)}. \end{aligned}$$

The result follows by straightforward linear algebra.  $\square$

**3.8** Note that if one specializes a curve  $C$  with a  $g_4^1$  to a trigonal curve, then actually  $C_{(4)} = \frac{1}{12}(2_*C - 6C + 2C^-) \approx_{alg} 0$ . However, we could not decide whether for a generic 4:1 cover of  $\mathbf{P}^1$  we have  $C_{(4)} \approx_{alg} 0$ .

## 4 Appendix

**4.1** We recall some facts on binomial coefficients. The binomial coefficients  $\binom{n}{k}$  are defined (also for negative  $n \in \mathbf{Z}$  !) as (cf. [ACGH], VIII.3)

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \quad (k > 0), \quad \binom{n}{0} := 1, \quad \binom{n}{k} := 0 \quad (k < 0).$$

With this definition, they are the coefficients in the expansion of  $(1+x)^n$ :

$$(1+x)^n = 1 + \binom{n}{1}x + \dots + \binom{n}{k}x^k + \dots = \sum_{k=0}^{\infty} \binom{n}{k}x^k.$$

Comparing the coefficients of  $x^k$  in  $(1+x)^n(1+x)^m = (1+x)^{n+m}$  one finds, for all  $n, m \in \mathbf{Z}$ :

$$\sum_i \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}, \quad (4.1.1)$$

**4.2 Lemma.** We have:

$$\sum_{i=0}^l (-1)^i \binom{m}{l-i} = \binom{m-1}{l}, \quad (4.2.1)$$

$$\sum_{i=0}^l (-1)^{i-1} i \binom{m}{l-i} = \binom{m-2}{l-1}, \quad (4.2.2)$$

$$\sum_{i=0}^l (-1)^i i^2 \binom{m}{l-i} = \binom{m-3}{l-2} - \binom{m-3}{l-1}. \quad (4.2.3)$$

**Proof.** Using  $\binom{-1}{k} = (-1)^k$  and 4.1.1, the first line can be written as:

$$\sum_{i=0}^l (-1)^i \binom{m}{l-i} = \sum_{i=0}^l \binom{-1}{i} \binom{m}{l-i} = \binom{m-1}{l}.$$

The second line follows in the same way, using  $\binom{-2}{k-1} = (-1)^{k-1}k$ :

$$\sum_{i=0}^l (-1)^{i-1} i \binom{m}{l-i} = \sum_{i=0}^l \binom{-2}{i-1} \binom{m}{l-i} = \binom{m-2}{l-1}.$$

For the last line we use  $\binom{-3}{k} = (-1)^k(k+1)(k+2)/2$ , so:

$$(-1)^k k^2 = 2\binom{-3}{k} - 3\binom{-2}{k} + \binom{-1}{k}.$$

Therefore:

$$\begin{aligned} \sum_{i=0}^l (-1)^i i^2 \binom{m}{l-i} &= 2 \sum_{i=0}^l \binom{-3}{i} \binom{m}{l-i} - 3 \sum_{i=0}^l \binom{-2}{i} \binom{m}{l-i} + \sum_{i=0}^l \binom{-1}{i} \binom{m}{l-i} \\ &= 2\binom{m-3}{l} - 3\binom{m-2}{l} + \binom{m-1}{l}. \end{aligned}$$

Now use that:

$$\begin{aligned} \binom{m-1}{l} &= \binom{m-2}{l-1} + \binom{m-2}{l} = \binom{m-3}{l-2} + 2\binom{m-3}{l-1} + \binom{m-3}{l}, \quad \text{and} \\ \binom{m-2}{l} &= \binom{m-3}{l-1} + \binom{m-3}{l}. \end{aligned}$$

□

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