On Hitchin's Connection.

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1 Introduction

1.1

The aim of this paper is to give an explicit expression for Hitchin's connection in the case of rank 2 bundles with trivial determinant over curves of genus 2.

We sketch the general situation. Let $\pi : \mathcal{C} \to S$ be a family of projective smooth curves. Let $p : \mathcal{M} \to S$ be the family of moduli spaces of (S-equivalence classes of) rank r bundles with trivial determinant on $\mathcal{C}$ over S. On $\mathcal{M}$ we have a naturally defined determinant linebundle $\mathcal{L}$. The vector bundles $p_*(\mathcal{L}^{\otimes k})$ on S have a natural (projective) flat connection (such a connection also exists for other structure groups besides $SL(r)$).

This connection first appeared in Quantum Field Theory (conformal blocks) and was subsequently studied by various mathematicians (see [Hi] and [BM] and the references given there). We follow the approach of Hitchin [Hi] who showed that this connection can be constructed along the lines of Welters' theory of deformations [W].

In case the curves have genus 2 and the rank $r = 2$, the family of moduli spaces $p : \mathcal{M} \to S$ is isomorphic (over S) to a $\mathbf{P}^3$-bundle

$$p : \mathcal{M} \cong \mathbf{P} \longrightarrow S.$$

Under this isomorphism $\mathcal{L}$ corresponds to $\mathcal{O}_{\mathbf{P}}(1) \otimes p^*\mathcal{N}$ for some line bundle $\mathcal{N}$ on S. Giving a projective connection on $p_*\mathcal{L}^{\otimes k}$ is then the same as giving a projective connection on $p_*\mathcal{O}_{\mathbf{P}}(k)$

$$\nabla^{(k)} : p_*\mathcal{O}_{\mathbf{P}}(k) \longrightarrow p_*\mathcal{O}_{\mathbf{P}}(k) \otimes \Omega^1_S.$$

1.2

In the first part of the paper (Section 2) we give a characterization of Hitchin's connection (strictly speaking, we define a new connection which is almost characterized by Hitchin's requirements).

We first develop a bit of general theory concerning heat operators. A heat operator is a certain type of map $D : p^*(\mathcal{T}_S) \to \mathrm{Diff}^{(2)}_{\mathcal{M}}(\mathcal{L}^{\otimes k})$. Thus a vector field θ on S gives a second order differential operator $D(\theta)$ on $\mathcal{L}^{\otimes k}$, which determines a map $D(\theta) : p_*\mathcal{L}^{\otimes k} \rightarrow p_*\mathcal{L}^{\otimes k}$. These then determine a connection on $p_*\mathcal{L}^{\otimes k}$. The symbol of D is a map $\rho_D : p^*(\mathcal{T}_S) \rightarrow S^2\mathcal{T}_{\mathcal{M}/S}$.

Hitchin showed that if a heat operator D determines the desired connection on $p_*\mathcal{L}^{\otimes k}$, then its symbol map:

$$\rho_D^s : p^*(\mathcal{T}_S) \longrightarrow S^2\mathcal{T}_{\mathcal{M}^s/S},$$

where $\mathcal{M}^s$ is the locus of stable bundles in $\mathcal{M}$, must be equal to a map $1/(k+2)\rho^s_{Hitchin}$. The map $\rho^s_{Hitchin}$ is determined by the deformation theory of a typical fibre C and of the stable bundles on C. He also proved that $\rho^s_{Hitchin}$ extends to a $\rho_{Hitchin} : p^*(\mathcal{T}_S) \rightarrow S^2\mathcal{T}_{\mathcal{M}/S}$ for bundles of rank r on curves of genus g, except maybe for the case $r = 2$, $g = 2$ (!) (in that case the non-stable bundles have codimension 1 in $\mathcal{M}$, whereas in all other cases they have codimension at least two). We show, using simple bundles, that $\rho^s_{Hitchin}$ also extends to a $\rho_{Hitchin}$ in that case (Theorem 2.6.1).

From a study of heat operators and their symbols we conclude that we can find a (projective) heat operator D with the right symbol over all of $\mathcal{M}$ (Corollary 2.6.3). Such a D need not be unique however.

The two-torsion group scheme $\mathcal{H} := Pic^0_{\mathcal{C}/S}[2]$ acts on $\mathcal{M}$ (tensoring a rank two bundle by a line bundle of order two doesn't change the determinant). Then $\mathcal{H}$ acts (projectively) on $\mathcal{L}^{\otimes k}$, this action can be linearized to the action of a central extension $\mathcal{G}$ of $\mathcal{H}$ on $\mathcal{L}^{\otimes k}$

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We show that there is a unique (projective) heat operator

$$\mathbf{D}_k : p^*(\mathcal{T}_S) \longrightarrow \mathrm{Diff}^{(2)}_{\mathcal{M}}(\mathcal{L}^{\otimes k}) \qquad \text{with} \quad \rho_{\mathbf{D}_k} = \tfrac{1}{k+2}\rho_{Hitchin}$$

and which commutes with the action of $\mathcal{G}$ (Proposition 2.6.4). The (projective) connection determined by $\mathbf{D}_k$ is called Hitchin's connection (cf. 2.6.6).

1.3

To be able to compute $\mathbf{D}_k$ (and to establish some of the results mentioned above), we use a map $f : \mathrm{Pic}^0_{\mathcal{C}/S} \rightarrow \mathcal{M}$ over S whose image is the family of Kummer surfaces $\mathcal{K} \rightarrow S$. We show that the pull-back of $\mathbf{D}_k$ along f determines the connection on $q_*f^*\mathcal{L}^{\otimes k}$ given by the heat equations on (abelian) theta functions. (The connection defined by $\mathbf{D}_k$ on $p_*\mathcal{L}^{\otimes k}$ does not pull-back to the 'abelian' connection however.)

We show that $\mathbf{D}_k$ is characterized by this property. Moreover, we express this property in terms of the (local) equation defining $\mathcal{K} \subset \mathcal{M}$ (2.6.7 ($Eq.$)).

1.4

To write down a completely explicit connection (on $p_*\mathcal{O}_{\mathbf{P}}(k)$) we must now choose a certain S and a family of curves $\mathcal{C}$ over S. We take $S = \mathcal{P}$, the configuration space of $2g+2$ points in

$\mathbf{C}^{2g+2}$ and take the obvious family of hyperelliptic curves over it:

$$\begin{array}{lcl} \mathcal{C} & & \supset \mathcal{C}_z: \quad y^2 = \prod_i (x - z_i) \\ \downarrow & & \quad\downarrow \\ \mathcal{P} & := \mathbf{C}^{2g+2} - \{(z_1, \ldots, z_{2g+2}): \ z_i \neq z_j \text{ if } i \neq j\}, & \ni z \end{array}$$

We are in the case $g = 2$, but the connections we define might be interesting for any g.

The pull-back of the bundle $p : \mathbf{P} \to \mathcal{P}$ along an unramified $2^4 : 1$-Galois cover $\tilde{\mathcal{P}} \to \mathcal{P}$ with group $H = (\mathbf{Z}/2\mathbf{Z})^4$ is trivial. The bundle $\mathbf{P}$ can thus obtained as a quotient of $\tilde{\mathcal{P}} \times \mathbf{P}^3$ by H (4.5.1). Therefore, locally (in the complex or the etale topology), the vector bundles $p_*\mathcal{O}_{\mathbf{P}}(k)$ and the trivial bundle $S_k \otimes \mathcal{O}_{\mathcal{P}}$ are canonically isomorphic, with $S_k = H^0(\mathbf{P}^3, \mathcal{O}(k))$, the vector space of homogenous polynomials of degree k in 4 variables. Thus it suffices to write down the connection on the trivial bundle which corresponds to Hitchin's connection on $p_*\mathcal{O}_{\mathbf{P}}(k)$.

This connection on $S_k \otimes \mathcal{O}_{\mathcal{P}}$ is given by

$$\nabla : S_k \otimes \mathcal{O}_{\mathcal{P}} \longrightarrow S_k \otimes \Omega_{\mathcal{P}}, \qquad \nabla(w \otimes f) = w \otimes \mathrm{d}f - \tfrac{1}{k+2} \sum_{i \neq j} M_{ij}(w) \otimes \frac{f \mathrm{d}z_i}{z_i - z_j},$$

for certain endomorphism $M_{ij} \in \mathrm{End}(S_k)$. We give two descriptions of the endomorphisms M_{ij}. One is based on the Lie algebra $so(2g+2)$ and its (half)-spin representations. We will in fact identify $S_k = S^k V(\omega_{g+1})$ where $V(\omega_{g+1})$ is a half spin representation of $so(6)$.

The other description uses the Heisenberg group action and we sketch that description here. The family $\mathcal{C}$ has $2g + 2$ sections which are the Weierstrass points on each fiber:

$$P_i : \mathcal{P} \longrightarrow \mathcal{C}, \qquad z \longmapsto (x, y) = (z_i, 0).$$

Each $P_i - P_j$ is a section in $\mathrm{Pic}^0_{\mathcal{C}/S}[2](S) \cong (\mathbf{Z}/2\mathbf{Z})^{2g}$. There is a central extension G of this group which acts on the S_k's (the groupscheme $\mathcal{G}$ is a twist of the constant group scheme G, these group schemes are isomorphic over $\tilde{P}$). Let $\pm U_{ij} \in \mathrm{End}(S_1)$ be the endomorphisms induced by $P_i - P_j$ with the property that U_{ij}^2 is the identity. Since $\mathrm{End}(S_1) = S_1 \otimes S_1^*$ and S_1^* may be identified with the derivations on $S := \oplus S_k$ and the composition $U_{ij}U_{ij}$ may be considered as a second order differential operator which acts on any S_k. The symbol (that is, the degree two part) of this operator is, up to a factor $-1/16$, the desired M_{ij} (which, in this sense, does not depend on k).

The idea for trying these Ω_{ij}'s comes from [vGP]. There the Hitchin map, a Hamiltonian system on the cotangent bundle $T^*\mathcal{M}^s$, is studied. The Hitchin map is basically the symbol of the heat operator $\mathcal{D}_k$ (cf. [Hi] §4). Among the results of [vGP] is an explicit description of the Hitchin map for $g = 2$ curves (obtained from line geometry in $\mathbf{P}^3$) and implicitly in that description is the one for the M_{ij} given here.

1.5

Now that the connection is determined it is interesting to consider its monondromy representation. In the case of rank two bundes, this has been considered by Kohno [K2]. We verify that our connection gives has the same local monodromy as Kohno's representation in Subsection 4.6. Kohno (and the physicists) use (locally on the base it seems) the Knizhnik-Zamolodzhikov equations (which define a flat connection over $\mathcal{P}$ using the Lie algebra $sl(2)$ [Ka], Chapter XIX). It is not clear to us how the KZ-equations are related to our connection (which is based on $so(2g+2)$). We do not know which representations of the braid group one obtains from our connections.

2 Characterizing Hitchin's connection

2.1 Introduction

We recall some deformation theory as described by Welters [W] and relate it to connections. Then we introduce heat operators and show how they define (projective) connections on certain bundles. Next we give a criterion for the existence of a heat equation with a given symbol and in Section 2.4 we recall the definition of the symbol of Hitchin's connection. In Section 2.5 we combine these results and give criteria (in 2.6.7 and 2.6.6) which determine Hitchin's connection.

2.2 Deformation theory

2.2.1 Deformations

Let X be a smooth projective variety over $\mathbf{C}$ with tangent bundle T_X. Let L be an invertible $\mathcal{O}_X$-module on X and let $s \in H^0(X, L)$. We are going to classify first order deformations of X, of the pair (X, L) and of the triple (X, L, s). See [W] for proofs.

Isomorphism classes of first order deformations X_ϵ of X (over $\operatorname{Spec}\mathbf{C}[\epsilon]/(\epsilon^2)$) are classified by $H^1(X, T_X)$. We write $[X_\epsilon]$ for the class in $H^1(X, T_X)$ of the deformation X_ϵ; similar notation will be used throughout.

The first order deformations (X_ϵ, L_ϵ) of the pair (X, L) are classified by $H^1(X, \mathrm{Diff}^{(1)}_X(L))$. Here $\mathrm{Diff}^{(1)}_X(L)$ is the sheaf of first order differential operators on L. This sheaf sits in an exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{Diff}^{(1)}_X(L) \stackrel{\sigma}{\longrightarrow} T_X \longrightarrow 0$$

where the map $\sigma : \mathrm{Diff}^{(1)}_X(L) \to T_X$ gives the symbol of the operator. This sequence gives a map $\alpha : H^1(X, \mathrm{Diff}^{(1)}_X(L)) \to H^1(X, T_X)$, which maps $[(X_\epsilon, L_\epsilon)]$ to $[X_\epsilon]$.

Finally, we consider first order deformations of the triple (X, L, s). The evaluation map $d^1 s : \mathrm{Diff}^{(1)}_X(L) \to L,\ D \mapsto Ds$ gives a complex:

$$0 \longrightarrow \mathrm{Diff}^{(1)}_X(L) \stackrel{d^1 s}{\longrightarrow} L \longrightarrow 0.$$

Let $\mathbf{H}^1(d^1 s)$ be the first hypercohomology group of this complex. This is the space classifying the isomorphism classes of deformations $(X_\epsilon, L_\epsilon, s_\epsilon)$ of (X, L, s) ([W], Prop. 1.2). The spectral sequence connecting hypercohomology to cohomology gives a map $\beta : \mathbf{H}^1(d^1 s) \to H^1(\mathrm{Diff}^{(1)}_X(L))$, which maps $[(X_\epsilon, L_\epsilon, s_\epsilon)]$ to $[(X_\epsilon, L_\epsilon)]$.

Any element of $\mathbf{H}^1(d^1s)$ may be represented by a Cech cocycle. Let us choose an affine open covering $\mathcal{U} : X = \bigcup U_i$ and write $U_{ij} = U_i \cap U_j$. A Cech cocycle is given by a pair $(\{t_i\}, \{D_{ij}\})$ satisfying the relations:

$$t_i - t_j = D_{ij}(s), \quad D_{jk} - D_{ik} + D_{ij} = 0 \qquad (t_i \in L(U_i),\ D_{ij} \in \mathrm{Diff}_X^{(1)}(L)(U_{ij})).$$

The maps β and $\alpha \circ \beta$ correspond to forgetting first s_ϵ and then L_ϵ. They can be described as follows. The cocycle $(\{t_i\}, \{D_{ij}\})$ is mapped to the 1-cocycle $\{D_{ij}\}$ in the sheaf $\mathrm{Diff}_X^{(1)}(L)$ under β. This is then mapped to the 1-cocycle $\{\mathrm{Diff}^{(1)}(D_{ij})\}$ in the sheaf T_X.

Let us make explicit a deformation $(X_\epsilon, L_\epsilon, s_\epsilon)$ associated to $(\{t_i\}, \{D_{ij}\})$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ij} = \mathrm{Spec}(A_{ij})$. Let $M_i = L(U_i)$ and $M_{ij} = L(U_{ij})$. The section s gives elements $s_i \in M_i$; note that $t_i \in M_i$ as well. Write $\theta_{ij} = \sigma(D_{ij}) \in T_X(U_{ij})$. The 1-cocycle $\{\theta_{ij}\}$ defines a scheme X_ϵ over $\mathrm{Spec}(\mathbf{C}[\epsilon])$ by glueing the rings $A_i[\epsilon]$ via the isomorphisms

$$A_{ij}[\epsilon] \longrightarrow A_{ij}[\epsilon], \qquad f + g\epsilon \longmapsto f + (\theta_{ij}(f) + g)\epsilon.$$

Here we view θ_{ij} as a derivation. The 1-cocycle $\{D_{ij}\}$ in $\mathrm{Diff}_X^{(1)}(L)$ gives a deformation L_ϵ of L by glueing the $A_i[\epsilon]$-modules $M_i[\epsilon]$ via the $A_{ij}[\epsilon]$-module isomorphisms:

$$\phi_{ij} : M_{ij}[\epsilon] \longrightarrow M_{ij}[\epsilon], \qquad m_{ij} + m'_{ij}\epsilon \longmapsto m_{ij} + (D_{ij}(m_{ij}) + m'_{ij})\epsilon.$$

Note one has $D_{ij}(fs) = fD_{ij}(s) + \theta_{ij}(f)s$ by definition of the symbol σ and $\sigma(D_{ij}) = \theta_{ij}$. The cocycle $(\{t_i\}, \{D_{ij}\})$ in $\mathbf{H}^1(d^1s)$ defines a deformation $s_\epsilon \in H^0(X_\epsilon, L_\epsilon)$. The section s_ϵ is given by the family of elements:

$$s_\epsilon := \{s_{\epsilon,i}\} \quad (\in M_i[\epsilon]) \qquad \text{with} \quad s_{\epsilon,i} := s_i + t_i\epsilon.$$

The cocycle relation $t_i - t_j = D_{ij}(s)$ shows that the $s_{\epsilon,i}$'s glue: $\phi_{ij}(s_{\epsilon,j}) = s_{\epsilon,i}$.

2.2.2 Second order operators

The following construction is useful to obtain elements in $\mathbf{H}^1(d^1s)$ ([W], (1.9)). Let $\mathrm{Diff}_X^{(2)}(L)$ be the sheaf of second order differential operators on L. The symbol gives an exact sequence which is the first row of the complex:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Diff}_X^{(1)}(L) & \longrightarrow & \mathrm{Diff}_X^{(2)}(L) & \longrightarrow & S^2T_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & L & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}$$

the vertical maps are evaluation on s $(D \mapsto Ds)$, thus the first column is the complex d^1s considered before. From the exact sequence of hypercohomology one obtains a map

$$\delta_s : H^0(S^2T_X) \longrightarrow \mathbf{H}^1(d^1s).$$

Let us give a description in terms of cocycles of this map. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X as before. Let $w \in H^0(X, S^2T_X)$. We can find operators $D_i^{(2)} \in \mathrm{Diff}_X^{(2)}(L)(U_i)$ which map to $w|_{U_i}$ in $S^2T_X(U_i)$. Then

$$\delta_s(w) = (\{D_i^{(2)}(s)\}, \{D_i^{(2)} - D_j^{(2)}\}) \qquad (\in \mathbf{H}^1(d^1s)).$$

It is easy to verify that $\delta_s(w)$ is indeed a cocycle for the complex defined by d^1s. Note $w = \sigma(D_i^{(2)}) = \sigma(D_j^{(2)})$ on U_{ij}, thus $D_i^{(2)} - D_j^{(2)}$ is a first order operator.

Note that $\beta(\delta_s(w)) = \{D_i^{(2)} - D_j^{(2)}\}$ $(\in \mathrm{Diff}_X^{(1)}(L))$ is *independent* of the choice of the section s. Therefore, given w $(\in H^0(X, S^2T_X))$ we get a deformation (X_ϵ, L_ϵ) such that *any* section $s \in H^0(X, L)$ is deformed to a section $s_\epsilon \in H^0(X_\epsilon, L_\epsilon)$.

2.2.3 Infinitesimal connections

Let (X_ϵ, L_ϵ) be a deformation of (X, L). We consider a differential operator D of order at most two on L_ϵ which locally can be written as (cf. 2.3.2 below)

$$D = \{D_i\} \quad (\in H^0(X_\epsilon, \mathrm{Diff}_{X_\epsilon}^{(2)}(L_\epsilon))), \qquad D_i = \partial_\epsilon + R_i : M_i[\epsilon] \longrightarrow M_i[\epsilon]$$

with $\partial_\epsilon(n + \epsilon m) = m$ for $n, m \in M_i$, and R_i a $\mathbf{C}[\epsilon]$-linear map of order at most 2 on M_i (so $s_i \mapsto R_i(fs_i) - fR_i(s_i)$ has order at most $k-1$ and a map of order zero is by definition an $A_i[\epsilon]$-linear map). Thus R_i is a second order differential operator involving only deriviatives in the fiber directions (and not in the base direction ϵ). The symbol of such a D is the second order part of the D_i (and thus of the R_i). The restriction of the symbol to X $(\subset X_\epsilon)$ is:

$$\bar{\rho}(D) := \{\sigma(\bar{R}_i)\} \quad (\in H^0(X, S^2T_X)), \qquad \text{where} \quad R_i = \bar{R}_i + \epsilon\bar{R}'_i$$

with $\bar{R}_i$, $\bar{R}'_i : M_i \to M_i[\epsilon]$ (recall R_i is ϵ-linear).

For any $s \in H^0(X, L)$ we then have a hyper cohomology class

$$\delta_s(w) := (\{\bar{R}_i(s_i)\}, \{\bar{R}_i - \bar{R}_j\}) \qquad (\in \mathbf{H}^1(d^1s)).$$

First of all we show that the deformation (X_ϵ, L_ϵ) is the deformation determined by $-\beta(\delta_s(\bar{\rho}(D)))$:

$$[(X_\epsilon, L_\epsilon)] + \beta(\delta_s(\bar{\rho}(D))) = 0 \qquad (\in H^1(X, \mathrm{Diff}_X^{(1)}(L))).$$

This can be verified as follows. Let $s_{\epsilon,i} \in M_i[\epsilon]$ such that $s_{\epsilon,i} = \phi_{ij}(s_{\epsilon,j})$. Then we have $\phi_{ij}(D_j(s_{\epsilon,j}) = D_i(\phi_{ij}(s_{\epsilon,j})$. With $s_{\epsilon,j} := s_j + t_j\epsilon$, $(s_j\, t_j \in M_j)$ this gives (for the 'constant' term):

$$D_{ij}(s_j) + t_j + \bar{R}_j(s_j) = t_j + \bar{R}_i(s_j),$$

which is zero for all such local sections iff

$$D_{ij} + (\bar{R}_j - \bar{R}_i) = 0 \qquad \in \mathrm{Diff}_X^{(1)}(L)(U_{ji}).$$

Since the $\bar{R}_j$ are local lifts of $\bar{\rho}(D)$, we get that $\beta\delta_s(\bar{\rho}(D)) = \{\bar{R}_i - \bar{R}_j\}$ $(\in \mathrm{Diff}_X^{(1)}(L)(U_{ij})$.

Next we consider any deformation $(X_\epsilon, L_\epsilon, s_\epsilon)$ of (X, L, ϵ). We say that a section $s_\epsilon = \{s_i + \epsilon t_i\} \in H^0(X_\epsilon, L_\epsilon)$ is flat for D if

$$D_i(s_i + \epsilon t_i) \equiv 0 \bmod (\epsilon), \qquad \text{equivalently} \quad t_i = -\bar{R}_i(s_i),$$

in that case we write $\nabla(D)(s_\epsilon) = 0$. Since $[(X_\epsilon, L_\epsilon, s_\epsilon)] = (\{t_i\}, \{D_{ij}\})$ $(\in \mathbf{H}(d^1 s))$ we get:

$$\delta_s(\bar{\rho}(D)) = -[(X_\epsilon, L_\epsilon, s_\epsilon)] \quad \text{iff} \quad \nabla(D)(s_\epsilon) = 0.$$

2.2.4 A remark on symbols

We use the conventions of [W] concerning differential operators and symbols. In particular when we write $S^2 T_X$ we mean symmetric tensors in the sheaf of vector fields on X. Note that if $X = \mathbf{A}^1_{\mathbf{C}}$ with coordinate t, then the symbol of the operator $\partial^2/\partial^2 t$ $(\in \mathrm{Diff}^{(1)}_{\mathcal{O}})$ is the symmetric section $2\partial/\partial t \otimes \partial/\partial t$ $(\in S^2 T_X)$. By abuse of notation we will sometimes write $\sigma(\partial^2/\partial^2 t) = 2\partial^2/\partial^2 t$.

2.3 Heat operators

2.3.1 Notation

Let $p : \mathcal{X} \to S$ be a smooth surjective morphism of smooth varieties over $\mathbf{C}$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{\mathcal{X}}$-module over $\mathcal{X}$.

We write $\mathcal{T}_{\mathcal{X}}$ (resp. $\mathcal{T}_S$, resp. $\mathcal{T}_{\mathcal{X}/S}$) for the sheaf of vector fields on $\mathcal{X}$ (resp. S, resp. $\mathcal{X}$ over S). We denote $\mathrm{Diff}^{(k)}_{\mathcal{X}}(\mathcal{L})$ the sheaf of differential operators of order at most k on $\mathcal{L}$ over $\mathcal{X}$. Further, $\mathrm{Diff}^{(k)}_{\mathcal{X}/S}(\mathcal{L}) \subset \mathrm{Diff}^{(k)}_{\mathcal{X}}(\mathcal{L})$ denotes the subsheaf of $p^{-1}(\mathcal{O}_S)$-linear operators. We remark that $\mathrm{Diff}^{(k)}_{\mathcal{X}}(\mathcal{L})$ is a coherent (left) $\mathcal{O}_{\mathcal{X}}$-module and that $\mathrm{Diff}^{(k)}_{\mathcal{X}/S}(\mathcal{L})$ is a coherent submodule. For any k there is a symbol map

$$\sigma^{(k)} : \mathrm{Diff}^{(k)}_{\mathcal{X}}(\mathcal{L}) \longrightarrow S^k \mathcal{T}_{\mathcal{X}} = \mathrm{Sym}^k_{\mathcal{O}_{\mathcal{X}}}(\mathcal{T}_{\mathcal{X}}).$$

It maps $\mathrm{Diff}^{(k)}_{\mathcal{X}/S}(\mathcal{L})$ into $S^k \mathcal{T}_{\mathcal{X}/S}$.

Consider a point $x \in \mathcal{X}(\mathbf{C})$, an (étale or analytic) neighbourhood U of $p(x)$ in S, and an (étale or analytic) neighbourhood V of x in $\mathcal{X}$ such that $p(V) \subset U$. Assume we have coordinate functions $t_1, \ldots, t_r \in \mathcal{O}_S(U)$ and functions $x_1, \ldots, x_n \in \mathcal{O}_{\mathcal{X}}(V)$ such that $t_1 = t_1 \circ p, \ldots, t_r = t_r \circ p, x_1, \ldots, x_n$ are coordinates on V. (In the étale case this means that $\Omega^1_U = \bigoplus \mathcal{O}_U \mathrm{d}t_i$ and $\Omega^1_{V/U} = \bigoplus \mathcal{O}_V \mathrm{d}x_i$.) Then the elements $\partial/\partial x_i$ form a basis for $\mathcal{T}_{\mathcal{X}/S}|_V$ over $\mathcal{O}_V$. The system $(V, t_1, \ldots, t_r, x_1, \ldots, x_n)$ will be called a *coordinate patch*. We can find coordinate patches around any point $x \in \mathcal{X}(\mathbf{C})$.

2.3.2 Definition

A *heat operator D on $\mathcal{L}$ over S* is a (left) $\mathcal{O}_{\mathcal{X}}$-module homomorphism

$$D : p^*(\mathcal{T}_S) \longrightarrow \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L})$$

which satisfies the following property: for any coordinate patch $(V, t_1, \ldots, t_r, x_1, \ldots, x_n)$ as above, and any trivialization $\mathcal{L}|_V \cong \mathcal{O}_V$, the operator $D(\partial/\partial t_1)$ has the following shape:

$$D(\frac{\partial}{\partial t_1}) = f + \frac{\partial}{\partial t_1} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{1 \leq i \leq j \leq n} f_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

for certain $f, f_i, f_{ij} \in \mathcal{O}_{\mathcal{X}}(V)$. More precisely, let

$$\mathcal{W}_{\mathcal{X}/S}(\mathcal{L}) = \mathrm{Diff}^{(1)}_{\mathcal{X}}(\mathcal{L}) + \mathrm{Diff}^{(2)}_{\mathcal{X}/S}(\mathcal{L}) \subset \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L})$$

be the subsheaf of second order differential operators on $\mathcal{L}$ whose symbol lies in $S^2(\mathcal{T}_{\mathcal{X}/S}) \subset S^2(\mathcal{T}_{\mathcal{X}})$. Note that $\mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L}) \subset \mathcal{W}_{\mathcal{X}/S}(\mathcal{L})$ and that there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L}) \longrightarrow \mathcal{W}_{\mathcal{X}/S}(\mathcal{L}) \longrightarrow p^*(\mathcal{T}_S) \oplus S^2\mathcal{T}_{\mathcal{X}/S} \longrightarrow 0.$$

Our condition on D above means that D is a map $D : p^*(\mathcal{T}_S) \to \mathcal{W}_{\mathcal{X}/S}(\mathcal{L})$ whose composition with the map $\mathcal{W}_{\mathcal{X}/S}(\mathcal{L}) \to p^*(\mathcal{T}_S)$ is the identity. (So $D(\partial/\partial t_1)$ has no terms involving $\partial/\partial t_i$ except for the term $\partial/\partial t_1$ with coefficient 1.)

Such a map D is determined by the $\mathcal{O}_S$-linear map $p_*D : \mathcal{T}_S \to p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))$. Note that $\mathcal{O}_S$ is a subsheaf of $p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))$. Any $\mathcal{O}_S$-linear map $\bar{D} : \mathcal{T}_S \to p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))/\mathcal{O}_S$ can locally be lifted to a map into $p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))$. This means that any point $s \in S(\mathbf{C})$ has a neighbourhood U such that there is an $\mathcal{O}_U$-linear map $D_U : \mathcal{T}_S|_U \to p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))|_U$ which reduces to $\bar{D}|_U$.

A *projective heat operator $\bar{D}$ on $\mathcal{L}$ over S* is an $\mathcal{O}_S$-linear map

$$\bar{D} : \mathcal{T}_S \longrightarrow p_*(\mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))\Big/\mathcal{O}_S$$

such that any local lifting D_U as above gives rise to a heat operator on $\mathcal{L}$ over U.

The *symbol* of a heat operator D is the $\mathcal{O}_{\mathcal{X}}$-linear map

$$\rho_D : p^*(\mathcal{T}_S) \longrightarrow S^2\mathcal{T}_{\mathcal{X}/S}$$

given by composing D with the symbol $\sigma^{(2)}$. We note that the symbol of a projective heat operator is well-defined.

2.3.3 Heat operators and connections

We claim that a heat operator D gives rise to a connection

$$\nabla(D) : p_*(\mathcal{L}) \longrightarrow p_*(\mathcal{L}) \otimes_{\mathcal{O}_S} \Omega^1_S.$$

Indeed, suppose that $s \in p_*(\mathcal{L})(U) = \mathcal{L}(p^{-1}(U))$ and that $\theta \in \mathcal{T}_S(U)$. Then $D(p^{-1}\theta)$ is a second order differential operator on $\mathcal{L}$ over $p^{-1}(U)$. Hence $D(p^{-1}\theta)(s) \in \mathcal{L}(p^{-1}(U)) = p_*(\mathcal{L})(U)$. By our local description of D above we see that $D(p^{-1}\theta)(fs) = fD(p^{-1}\theta)(s) + \theta(f)s$ for every $f \in \mathcal{O}_S(U)$. Hence the operation $(\theta, s) \mapsto D(p^{-1}\theta)(s)$ defines a connection $\nabla(D)$ on $p_*\mathcal{L}$ over S. In the sequel we will often write $D(\theta)$ in stead of $D(p^{-1}\theta)$.

If we have a projective heat operator $\bar{D}$, the connection $\nabla(\bar{D}) = \nabla(D_U)$ is only defined locally, by choosing lifts D_U. The difference of two local lifts is given by a local section $\eta \in \Omega^1_S(U)$, in which case the difference of the two connections $\nabla(D_U)$ is multiplication by η. Thus $\bar{D}$ defines unambiguously a projective connection $\bar{\nabla}(\bar{D})$.

2.3.4 Heat operators and deformations

Let D be a heat operator on $\mathcal{L}$ over S. Let $0 \in S(\mathbf{C})$ be a point of the base. Put $X = \mathcal{X}_0$ and $L = \mathcal{L}_0$. An element $\theta \in T_0 S$ will also be considered as a morphism $\theta : \mathrm{Spec}(\mathbf{C}[\epsilon]) \to S$. By base change we get a pair (X_ϵ, L_ϵ) over $\mathrm{Spec}(\mathbf{C}[\epsilon])$. Clearly, $w_\theta := \rho_D(\theta)$ is an element of $H^0(X, S^2 T_X)$. It can be seen by working through the definitions that $[(X_\epsilon, L_\epsilon)] + \beta(\delta_s(w_\theta)) = 0$ for any section $s \in H^0(X, L)$. (Note that in Subsection 2.2.2 we showed that $\beta(\delta_s(w_\theta))$ was independent of s.) Let $s_\epsilon \in H^0(X_\epsilon, L_\epsilon)$ lift s. One can show that $[(X_\epsilon, L_\epsilon, s_\epsilon)] = -\delta_s(w_\theta)$ if and only if the section s_ϵ is horizontal for the connection $\nabla(D)$.

2.3.5 Projective heat operators and change of line bundle

Let D be a heat operator on $\mathcal{L}$ over S. Let $g \in H^0(S, \mathcal{O}_S^*)$ be an invertible function on S. Multiplication by $p^{-1}(g)$ defines an invertible operator on $\mathcal{L}$, also denoted by g. For any local section θ of $\mathcal{T}_S$, we have the obvious relation

$$g^{-1} \circ D(\theta) \circ g = g^{-1}\theta(g) + D(\theta).$$

We conclude that $g^{-1} \circ D \circ g$ is a heat operator and that the projective heat operators $\bar{D}$ and $\overline{g^{-1} \circ D \circ g}$ are equal. (The difference is given by the section $g^{-1}\mathrm{d}g = \mathrm{d}\log g$ of Ω_S^1.)

Suppose that $\mathcal{M}$ is an invertible $\mathcal{O}_S$-module on S. The above implies that the set of projective heat operators on $\mathcal{L}$ over S can be identified canonically with the set of projective heat operators on $\mathcal{L}' = \mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} p^*(\mathcal{M})$. Indeed, choose a covering of S on whose members U_i the line bundle $\mathcal{M}$ becomes trivial. This identifies the heat operators on $\mathcal{L}$ and $\mathcal{L}'$ over U. The difference in the local identifications is given by the 1-forms $\mathrm{d}\log g_{ij} \in \Omega^1_{U_{ij}}$.

2.3.6 Heat operators and flatness

We say that a heat operator D on $\mathcal{L}$ over S is *flat* if $D([\theta, \theta']) = [D(\theta), D(\theta')]$ for any two local sections θ, θ' of $\mathcal{T}_S$ on $U \subset S$. It suffices to consider local vector fields θ, θ' on S with $[\theta, \theta'] = 0$ and to check that $[D(\theta), D(\theta')] = 0$. We remark that in this case the operator $[D(\theta), D(\theta')]$ is a section of $\mathrm{Diff}^{(3)}_{\mathcal{X}/S}(\mathcal{L})$ over $p^{-1}(U)$. We say that D is *projectively flat* if we have $D([\theta, \theta']) = h + [D(\theta), D(\theta')]$ for some function $h = h_{\theta,\theta'} \in \mathcal{O}_S(U)$ for any $\theta, \theta' \in \mathcal{T}_S(U)$. A projective heat operator $\bar{D}$ is called *projectively flat* if any of the local lifts D_U are projectively flat.

We remark that if the heat operator D is flat, then the associated connection $\nabla(D)$ on $p_*\mathcal{L}$ is flat as well. In particular, if $p_*\mathcal{L}$ is a coherent $\mathcal{O}_S$-module, then this implies that $p_*\mathcal{L}$ is locally free. In the same vein we have that a projectively flat projective heat operator $\bar{D}$ defines a projectively flat projective connection $\bar{\nabla}(\bar{D})$ on $p_*\mathcal{L}$.

2.3.7 Heat operators with given symbol

Let $\mathcal{X} \to S$ and $\mathcal{L}$ be as above, and assume that we are given a $\mathcal{O}_\mathcal{X}$-linear map $\rho : p^*\mathcal{T}_S \to S^2\mathcal{T}_{\mathcal{X}/S}$. The question we are going to study is the following:

When can we find a (projective) heat operator D on $\mathcal{L}$ over S, $D : p^*\mathcal{T}_S \to \mathcal{W}_{\mathcal{X}/S}$ $(\subset \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}))$ with $\rho_D = \rho$?

So the symbol ρ_D (which is the compsition of D with the map $\sigma^{(2)} : \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L})) \to S^2\mathcal{T}_{\mathcal{X}/S})$ should be equal to the given ρ.

Let us define a number of canonical maps associated to the situation. First we have the standard exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/S} \longrightarrow \mathcal{T}_{\mathcal{X}} \longrightarrow p^*\mathcal{T}_S \longrightarrow 0.$$

This gives us a $\mathcal{O}_S$-linear map (the *Kodaira-Spencer map*)

$$\kappa_{\mathcal{X}/S} \quad : \quad \mathcal{T}_S \longrightarrow R^1p_*\mathcal{T}_{\mathcal{X}/S}.$$

Next we have the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/S} \longrightarrow \mathrm{Diff}^{(2)}_{\mathcal{X}/S}(\mathcal{L})/\mathcal{O}_{\mathcal{X}} \longrightarrow S^2\mathcal{T}_{\mathcal{X}/S} \longrightarrow 0.$$

This gives an $\mathcal{O}_S$-linear map

$$\mu_{\mathcal{L}} \quad : \quad p_*S^2\mathcal{T}_{\mathcal{X}/S} \longrightarrow R^1p_*\mathcal{T}_{\mathcal{X}/S}.$$

The invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ is given by an element of $H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$. Using the map $\mathrm{d}\log : \mathcal{O}^*_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/S}$ we get an element of $H^1(\mathcal{X}, \Omega^1_{\mathcal{X}/S})$, and hence a section $[\mathcal{L}]$ in $H^0(S, R^1p_*(\Omega^1_{\mathcal{X}/S}))$. This will be called the cohomology class of $\mathcal{L}$. By cupproduct with $[\mathcal{L}]$ we get another map $p_*S^2\mathcal{T}_{\mathcal{X}/S} \to R^1p_*\mathcal{T}_{\mathcal{X}/S}$, $w \mapsto w \cup [\mathcal{L}]$. (There is a natural pairing $S^2\mathcal{T}_{\mathcal{X}/S} \otimes \Omega^1_{\mathcal{X}/S} \to \mathcal{T}_{\mathcal{X}/S}$.) It is shown in [W, 1.16] that we have $\mu_{\mathcal{L}}(w) = -w \cup [\mathcal{L}] + \mu_{\mathcal{O}}(w)$.

Now let θ be a local vector field on S, i.e., $\theta \in \mathcal{T}_S(U)$, with U affine. We want to find a $D(\theta) \in \mathcal{W}_{\mathcal{X}/S}(\mathcal{L})$ with symbol $\rho(\theta)$. Recall that $\mathcal{W}_{\mathcal{X}/S}(\mathcal{L})$ is an extension:

$$0 \longrightarrow \mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L}) \longrightarrow \mathcal{W}_{\mathcal{X}/S}(\mathcal{L}) \longrightarrow p^*(\mathcal{T}_S) \oplus S^2\mathcal{T}_{\mathcal{X}/S} \longrightarrow 0.$$

The obstruction against finding a lift $D(\theta)$ $(\in \mathcal{W}_{\mathcal{X}/S}(\mathcal{L}))$ of the section $p^{-1}(\theta) \oplus \rho(p^{-1}\theta)$ of $p^*\mathcal{T}_S \oplus S^2\mathcal{T}_{\mathcal{X}/S}$ over $p^{-1}(U)$ is an element $o(\theta, \mathcal{L})$ in $H^1(p^{-1}(U), \mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L})) = H^0(U, R^1p_*(\mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L})))$ (recall that U was assumed affine). There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L}) \longrightarrow \mathcal{T}_{\mathcal{X}/S} \longrightarrow 0.$$

We remark that the image of the class $o(\theta, \mathcal{L})$ in $R^1p_*\mathcal{T}_{\mathcal{X}/S}(U)$ is the section $\kappa_{\mathcal{X}/S}(\theta) + \mu_{\mathcal{L}}(\rho(\theta))$. The first condition that has to be satisfied for $\bar{D}$ to exist is therefore:

$$(*) \qquad \kappa_{\mathcal{X}/S} + \mu_{\mathcal{L}} \circ \rho = 0.$$

If this condition is satisfied, then we can find a lift of $p^{-1}(\theta) \oplus \rho(p^{-1}\theta)$ to an element $\overline{D}_\theta$ in the sheaf $\mathcal{W}_{\mathcal{X}/S}(\mathcal{L})/\mathcal{O}_{\mathcal{X}}$. This element is well defined up to addition of a section of $\mathcal{T}_{\mathcal{X}/S}$. Hence we get a second obstruction in

$$(**) \qquad \mathrm{Coker}\Big(p_*\mathcal{T}_{\mathcal{X}/S} \xrightarrow{[\mathcal{L}]\cup} R^1p_*\mathcal{O}_X\Big)$$

Let us assume for the moment that both obstructions vanish for any θ as above. Then we can locally find a $D(\theta)$ lifting the element $p^{-1}(\theta) \oplus \rho(p^{-1}\theta)$. Thus, if we have a basis $\theta_1, \ldots, \theta_r$ of $\mathcal{T}_S$ over U, then we can define D_U by the formula $D_U(\sum a_i\theta_i) = \sum a_i D(\theta_i)$ for certain choices of the elements $D(\theta_i)$. If there is some way of choosing the elements $D(\theta)$ uniquely up to an element of $p^{-1}\mathcal{O}_S(U)$, e.g. if the map in $(**)$ is injective and $p_*(\mathcal{O}_\mathcal{X}) = \mathcal{O}_S$, then D_U determines a unique projective heat operator over U. In this case the map ρ determines a unique projective heat operator on $\mathcal{L}$ over S.

Here is another set of hypotheses which imply the existence of heat operators with given symbol. Let $0 \in S(\mathbf{C})$ be a point. Put $X = \mathcal{X}_0$ and $L = \mathcal{L}_0$ as in 2.3.4. Let $\theta \in T_0S$ and consider (X_ϵ, L_ϵ) over $\mathrm{Spec}(\mathbf{C}[\epsilon])$ as in 2.3.4. We know from 2.3.4 that $[(X_\epsilon, L_\epsilon)] + \beta(\delta_s(\rho(\theta))) = 0$ if a heat operator exists. Suppose that

1. $R^1p_*\mathcal{O}_\mathcal{X}$ and $R^1p_*\mathcal{T}_{\mathcal{X}/S}$ are locally free on S, and have fibres $H^1(X, \mathcal{O}_X)$ and $H^1(X, T_X)$ at any point 0 in $S(\mathbf{C})$.

2. for any 0 and θ as above, we have $[(X_\epsilon, L_\epsilon)] + \beta(\delta_s(\rho(\theta))) = 0$.

If these conditions are satisfied, then we can at least find local liftings D_U of ρ to a heat operator. Again, we will get a "canonical" projective heat operator if there is a "preferred" way of choosing the local lifts $D(\theta_i)$.

2.3.8 Heat operators on abelian schemes

In this subsection, we let $\mathcal{X} \to S$ be an abelian scheme with zero section $o : S \to \mathcal{X}$. We assume the line bundle $\mathcal{L}$ is relatively ample on $\mathcal{X}$ over S, i.e., $\mathcal{L}$ defines a polarization of $\mathcal{X}$ over S. We will show that there exists a unique projective heat operator on $\mathcal{L}$ over S, see [W].

We first remark that in this case the map $(**)$ of 2.3.7 is an isomorphism. This implies that $\Gamma(p^{-1}(U), \mathrm{Diff}^{(1)}_{\mathcal{X}/S}(\mathcal{L})) = \mathcal{O}_S(U)$ for any $U \subset S$. Hence, by the discussions in 2.3.7 we get a unique heat operator as soon as we have a (symbol) map ρ satisfying $(**)$.

We have $\mu_\mathcal{O} = 0$ in this situation, as we can let elements of $S^2\mathcal{T}_{\mathcal{X}/S}$ act by translation invariant operators (see [W, 1.20]). Hence, $\mu_\mathcal{L}(-) = -[\mathcal{L}] \cup (-)$. Now we use that

$$p_*(\otimes^2\mathcal{T}_{\mathcal{X}/S}) \xrightarrow{[\mathcal{L}]\cup} R^1p_*\mathcal{T}_{\mathcal{X}/S} \cong o^*\mathcal{T}_{\mathcal{X}/S} \otimes R^1p_*\mathcal{O}_\mathcal{X}$$

is an isomorphism. Consider $\rho := ([\mathcal{L}]\cup)^{-1} \circ \kappa_{\mathcal{X}/S}$. It is well known that this is a map into $p_*S^2\mathcal{T}_{\mathcal{X}/S}$ and it solves $(*)$ by definition; it is of course the unique solution to $(*)$. The unique projective heat operator $\bar{D} = \bar{D}_\mathcal{L}$ with $\rho = \rho_{\bar{D}}$ is called the projective heat operator associated to $(\mathcal{X}/S, \mathcal{L})$.

We make the remark that if $\mathcal{L}' = \mathcal{L}^{\otimes n}$, then we have $\rho_{\bar{D}'} = (1/n)\rho_{\bar{D}}$. It is known that these projective heat operators are projectively flat. This can be seen in a number of ways; one way is to show that $p_*(\mathrm{Diff}^{(3)}_{\mathcal{X}/S}(\mathcal{L})) = \mathcal{O}_S$ in this case, compare 2.3.6.

Uniqueness of the construction of $\bar{D}$ implies that $\bar{D}$ commutes (projectively) with the action of the theta-group $\mathcal{G}(\mathcal{L})$. In particular the projective monodromy group of the (projectively flat) connection $\bar{\nabla}(\bar{D})$ on $(p_*\mathcal{L})_0$ normalizes the action of $\mathcal{G}(L)$.

2.3.9 Heat operators and functoriality

Let $\mathcal{X} \to S$ and $\mathcal{L}$ as above, and let D be a heat operator on $\mathcal{L}$ over S. Note that for any morphism of smooth schemes $\varphi : S' \to S$ there is a pullback heat operator $D' = \varphi^*(D)$ on the pull back $\mathcal{L}'$ on $\mathcal{X}' = S' \times_S \mathcal{X}$. We leave it to the reader to give the precise definition. This pullback preserves flatness. If $\varphi^*(p_*\mathcal{L}) \cong p'_*\mathcal{L}'$, then the connection $\nabla(D')$ is the pullback of the connection $\nabla(D)$. In addition pullback is well defined for projective heat operators and preserves projective flatness.

If $\psi : \mathcal{U} \to \mathcal{X}$ is an étale morphism of schemes over S, then the heat operator D induces a heat operator $\psi^*(D)$ on $\psi^*(\mathcal{L})$ over S.

2.3.10 Heat operators and compatibility

Suppose we have a second smooth surjective morphism $q : \mathcal{Y} \to S$ and that we have a morphism $f : \mathcal{Y} \to \mathcal{X}$ over S. We will assume that f is a submersion. This means that for any point $y \in \mathcal{Y}(\mathbf{C})$ there is a coordinate patch $(V, t_1, \dots, t_r, x_1, \dots, x_c, y_1, \dots, y_m)$ of $f(y)$ in $\mathcal{X}$ such that $(f^{-1}(V), t_1, \dots, t_r, y_1 = y_1 \circ f, \dots, y_m = y_m \circ f)$ is a coordinate patch for y on $\mathcal{Y}$ and $x_i \circ f = 0$.

Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ as before. Let us say that a (projective) heat operator D on $\mathcal{L}$ over S is *(weakly) compatible* with $f : \mathcal{Y} \to \mathcal{X}$ if whenever we have a coordinate patch $(V, t_1, \dots, t_r, x_1, \dots, x_c, y_1, \dots, y_m)$ as above, and a trivialization $\mathcal{L}|_V \cong \mathcal{O}_V$, then the operator $D(\partial/\partial t_1)$ has the following symbol:

$$\rho_D(\frac{\partial}{\partial t_1}) = \sum\nolimits_{i \le j=1}^{m} f_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum\nolimits_{i=1}^{c} x_i \, \Xi_i$$

for certain $f_{ij} \in \mathcal{O}_{\mathcal{X}}(V)$ and $\Xi_i \in S^2\mathcal{T}_{\mathcal{X}/S}(V)$. More precisely, this means there exists a $\rho = \rho_{f,D} : q^*\mathcal{T}_S \to S^2\mathcal{T}_{\mathcal{Y}/S}$ such that the following diagram commutes:

$$\begin{array}{ccccc} f^*(\rho_D) & : & f^*p^*\mathcal{T}_S & \longrightarrow & f^*S^2\mathcal{T}_{\mathcal{X}/S} \\ & & || & & \uparrow \\ \rho_{f,D} & : & q^*\mathcal{T}_S & \longrightarrow & S^2\mathcal{T}_{\mathcal{Y}/S} \end{array}$$

Suppose that D' is a heat operator on $f^*\mathcal{L}$ on $\mathcal{Y}$ over S. We say that D' is *(weakly) compatible* with D if D is (weakly) compatible with f and we have $\rho_{D'} = \rho_{f,D}$. In this case it is not true in general that $D(\theta)(s)|_{\mathcal{Y}} = D'(\theta)(s|_{\mathcal{Y}})$.

Assume that D is compatible with f. Let us write $N_{\mathcal{Y}}\mathcal{X}$ for the generalized normal bundle of $\mathcal{Y}$ in $\mathcal{X}$, i.e., $N_{\mathcal{Y}}\mathcal{X} = \mathrm{Coker}(\mathcal{T}_{\mathcal{Y}/S} \to f^*\mathcal{T}_{\mathcal{X}/S})$. Choose a local coordinate patch $(f^{-1}(V), t_1, \dots, t_r, y_1 = y_1 \circ f, \dots, y_m = y_m \circ f)$ as above. Write the operator $D(\partial/\partial t_1)$ in the form

$$D(\frac{\partial}{\partial t_1}) = f + \frac{\partial}{\partial t_1} + \sum\nolimits_{i=1}^{c} f_i \frac{\partial}{\partial x_i} + \sum\nolimits_{i=1}^{m} g_i \frac{\partial}{\partial y_i} + \sum\nolimits_{i \le j=1}^{m} f_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum\nolimits_{i=1}^{c} x_i \, \Xi_i,$$

for certain local functions f, f_i, g_i, f_{ij} and second order operators Ξ_i. It can be seen (and we leave this to the reader) that the class of $\sum f_i \frac{\partial}{\partial x_i}$ in $N_{\mathcal{Y}}\mathcal{X}(f^{-1}(V))$ is independent of the choice of the local trivialization and coordinates. The upshot of this is that if D is compatible with $\mathcal{Y} \to \mathcal{X}$, then there is a first order symbol

$$\sigma_{f,D} : q^*\mathcal{T}_S \longrightarrow N_{\mathcal{Y}}\mathcal{X}.$$

We say that D is *strictly compatible* with $f : \mathcal{Y} \to \mathcal{X}$ if D is compatible with f and the symbol $\sigma_{f,D}$ is zero. We remark that this notion is well defined for a projective heat operator as well.

Assume that D is strictly compatible with $\mathcal{Y} \to \mathcal{X}$. It follows from the local description of our compatibility of restrictions that D induces a heat operator $D_{\mathcal{Y}}$ on the invertible $\mathcal{O}_{\mathcal{Y}}$-module $f^*\mathcal{L}$ over S. It is characterized by the property $D(\theta)(s)|_{\mathcal{Y}} = D_{\mathcal{Y}}(\theta)(s|_{\mathcal{Y}})$ for any local section θ of $\mathcal{T}_S$ and any local section s of $\mathcal{L}$. It is clear that the symbol of $D_{\mathcal{Y}}$ is equal to $\rho_{f,D}$, hence that $D_{\mathcal{Y}}$ is compatible with D. We say that a heat operator D' on $f^*\mathcal{L}$ is *strictly compatible* with D if D is strictly compatible with $f : \mathcal{Y} \to \mathcal{X}$ and $D' = D_{\mathcal{Y}}$. In this case the restriction map

$$p_*\mathcal{L} \longrightarrow q_*f^*\mathcal{L}$$

is horizontal for the connections induced by D and $D' = D_{\mathcal{Y}}$. Note $D_{\mathcal{Y}}$ is flat if D is flat. Similar remarks hold for projective heat operators and projective flatness.

2.4 Moduli of bundles

2.4.1 The symbol $\rho_{Hitchin}$ in terms of moduli of bundles

In this subsection we explain how to get a symbol map ρ as in 2.3.7 in the case of the relative moduli space of rank 2 bundles of a family of curves.

Let C be a smooth projective curve over $\mathrm{Spec}(\mathbf{C})$, and let E be a stable invertible $\mathcal{O}_C$-module of rank 2 with $\det(E) \cong \mathcal{O}_C$. This gives a point $[E] \in \mathcal{M}_C(\mathbf{C})$. There are canonical identifications

$$T_{[E]}\mathcal{M}_C = H^1(C, \mathcal{E}nd_0(E)) \qquad \text{and} \quad T^*_{[E]}\mathcal{M}_C = H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1_C)$$

where $\mathcal{E}nd_0(E)$ denotes the sheaf of endomorphisms of E with trace zero. A cotangent vector $\phi \in T^*_{[E]}\mathcal{M}_C$ thus corresponds to a homomorphism of $\mathcal{O}_C$-modules $\phi : E \to E \otimes \Omega^1_C$. (Note that this is a Higgs field on the bundle E.) Composing and taking the trace gives a symmetric bilinear pairing

$$\begin{array}{ccc} T^*_{[E]}\mathcal{M}_C \times T^*_{[E]}\mathcal{M}_C & \longrightarrow & H^0(C, (\Omega^1_C)^{\otimes 2}) \\ (\phi, \psi) & \longmapsto & \mathrm{Trace}(\phi \circ \psi). \end{array}$$

We dualize this and use Serre duality to obtain a map

$$\rho_{C,E} : H^1(C, T_C) \longrightarrow S^2 T_{[E]}\mathcal{M}_C.$$

Let us make the observation that the construction above can be performed in families. Let $\pi : \mathcal{C} \to S$ a smooth projective family of curves over the scheme S smooth over $\mathrm{Spec}(\mathbf{C})$. Let $p : \mathcal{M} \to S$ be the associated family of moduli spaces of rank 2 semi-stable bundles with trivial determinant up to S-equivalence. We remark that p is a flat projective morphism, not smooth in general. However, the open part of stable bundles $\mathcal{M}^s \subset \mathcal{M}$ is smooth over S. We denote this smooth morphism by $p^s : \mathcal{M}^s \to S$.

Now let $T \to S$ be a morphism of finite type. We denote by an index $_T$ base change to T. Let $\mathcal{E}$ be a locally free sheaf of rank 2 on $\mathcal{C}_T$. For a point $0 \in T(\mathbf{C})$ we put $C = \mathcal{C}_0$ and $E = \mathcal{E}_0 = \mathcal{E}|_C$. Suppose that for any 0 we have (a) $\det(E) \cong \mathcal{O}_C$, and (b) the bundle E is stable. In this case we get an S-morphism $t : T \to \mathcal{M}^s$. Analogously to the above we have a canonical isomorphism

$$t^*(\Omega^1_{\mathcal{M}^s/S}) \cong (\pi_T)_*(\mathcal{E}nd_0(\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{C}_T}} \Omega^1_{\mathcal{C}_T/T}).$$

Using the same pairing and dualities as above we get an $\mathcal{O}_T$-linear map

$$R^1\pi_*\mathcal{T}_{\mathcal{C}_T/T} \longrightarrow S^2\Big(t^*(\mathcal{T}^1_{\mathcal{M}^s/S})\Big).$$

If we compose this with the pullback to T of the Kodaira-Spencer map $\mathcal{T}_S \to R^1\pi_*\mathcal{T}_{\mathcal{C}/S}$ of the family $\mathcal{C} \to S$, then we get a map

$$\rho_{\mathcal{E}/\mathcal{C}_T} : \mathcal{T}_S \otimes \mathcal{O}_T \longrightarrow S^2\Big(t^*(\mathcal{T}^1_{\mathcal{M}^s/S})\Big).$$

If there existed a universal bundle $\mathcal{E}^{univ}$ over $\mathcal{M}^s$, then we would get a "symbol"

$$\rho^s_{Hitchin} = \rho_{\mathcal{E}^{univ}} : (p^s)^*\mathcal{T}_S \to S^2\mathcal{T}_{\mathcal{M}^s/S}.$$

Although the universal bundle does not exist, the symbol $\rho^s_{Hitchin}$ does: it is the unique $\mathcal{O}_{\mathcal{M}^s}$-linear map such that, whenever $\mathcal{E}/\mathcal{C}_T$ is given, the pullback of $\rho_{Hitchin}$ to T agrees with $\rho^s_{\mathcal{E}/\mathcal{C}_T}$. Uniqueness and existence follows from the existence of "universal" bundles étale locally over $\mathcal{M}^s$ and the invariance of $\rho_{\mathcal{E}/\mathcal{C}_T}$ under automorphisms of $\mathcal{E}$ over $\mathcal{C}_T$.

2.4.2 Hitchin's results in genus at least 3

Let us get back to our curve C over $\mathrm{Spec}(\mathbf{C})$. We see from the above that any deformation C_ϵ of the curve will give a tensor in $\Gamma(\mathcal{M}^s_C, S^2\mathcal{T}_{\mathcal{M}_C})$. However, since $\mathcal{M}_C$ has points which do not correspond to stable bundles (and these points are singular points of $\mathcal{M}_C$ for $g > 2$) such a deformation does not give a global tensor in $H^0(\mathcal{M}_C, S^2T_{\mathcal{M}_C})$. In [Hi, Section 5], one can find a sketch of a proof that the tensor extends if $g > 2$ (or rank > 2, a case we do not even consider, although our present considerations work in that case as well).

In the situation $\mathcal{C} \to S$ there is a canonical line bundle $\mathcal{L}$ on $\mathcal{M}$; it can be defined by the formula

$$\mathcal{L} = \det R(pr_2)_*\mathcal{E}^{univ}, \qquad pr_2 : \mathcal{C} \times_S \mathcal{M} \to \mathcal{M}.$$

It is shown in [Hi, Theorem 3.6, Section 5], that if $g > 2$ the tensor $2/(2k + \lambda)\rho_{Hitchin}$ is the symbol of a unique projective flat heat operator $\bar{D}_{Hitchin}$ on $\mathcal{L}^{\otimes k}$ over S. (We remark that the extra factor 2 comes from our way of normalizing symbols, see Subsection 2.2.4.) In fact the arguments of [Hi] prove the existence of this heat operator over the moduli space of curves of genus $g > 2$.

We will not use these results.

2.5 The case of genus two curves

2.5.1 Notation in genus 2 case

We fix a scheme S smooth over $\mathrm{Spec}(\mathbf{C})$ and a smooth projective morphism $\pi : \mathcal{C} \to S$, whose fibres C are curves of genus 2. We write $0 \in S(\mathbf{C})$ for a typical point and $C = \mathcal{C}_0$.

We introduce the following objects associated to the situation. $\mathrm{Pic}^1 = \mathrm{Pic}^1_{\mathcal{C}/S} \to S$ denotes the Picard scheme of invertible $\mathcal{O}_{\mathcal{C}}$ bundles of relative degree 1 on $\mathcal{C}$ over S. There is a natural morphism $\mathcal{C} \to \mathrm{Pic}^1$ over S, given by $P \in C(\mathbf{C})$ maps to $[\mathcal{O}_C(P)]$ in $(\mathrm{Pic}^1)_0(\mathbf{C}) = \mathrm{Pic}^1(C)$. The image of this morphism is a relative divisor $\Theta^1 = \Theta^1_{\mathcal{C}/S}$ on Pic^1 over S; of course $\Theta^1 \cong \mathcal{C}$ as S-schemes. Let us denote $\alpha : \mathrm{Pic}^1 \to S$ the structural morphism. Note that $\alpha_*\mathcal{O}_{\mathrm{Pic}^1}(2\Theta^1)$ is a locally free $\mathcal{O}_S$-module of rank 4 on S. We put $\mathbf{P} = |2\Theta^1|$ equal to the projective space of lines in this locally free sheaf. More precisely, we define

$$p : \mathbf{P} = \mathbf{P}\Big(\mathcal{H}om_{\mathcal{O}_S}\big(\alpha_*\mathcal{O}_{\mathrm{Pic}^1}(2\Theta^1)\ ,\ \mathcal{O}_S\big)\Big) \longrightarrow S,$$

see [Ha, page 162] for notation used. For a point $0 \in S(\mathbf{C})$ we put $P = \mathbf{P}_0$.

Let $T \to S$ be a morphism of finite type, and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_{\mathcal{C}_T}$-modules of rank 2. For a point $0 \in T(\mathbf{C})$ we put $C = \mathcal{C}_0$ and $E = \mathcal{E}_0 = \mathcal{E}|_C$. Suppose that for any 0 we have

(a) $\det(E) \cong \mathcal{O}_C$,
(b) the bundle E is *semi*-stable.

(Compare with Subsection 2.4.1.) We recall the relative divisor $\mathcal{D}_{\mathcal{E}} \subset \mathrm{Pic}^1_T$ associated to $\mathcal{E}$. Set-theoretically it has the following description:

$$D_E := \mathcal{D}_{\mathcal{E}} \cap \mathrm{Pic}^1(C) = \{[L] \in \mathrm{Pic}^1(C) : \dim H^1(C, E \otimes L) \geq 1\}$$

To construct $\mathcal{D}_{\mathcal{E}}$ as a closed subscheme we may work étale locally on T. Hence we may assume that a Poincaré line bundle $\mathcal{L}$ on $\mathcal{C}_T \times_T \mathrm{Pic}^1_T$ exists, i.e., which has relative degree 1 for $p_2 : \mathcal{C}_T \times_T \mathrm{Pic}^1_T \to \mathrm{Pic}^1_T$ and induces $\mathrm{id} : \mathrm{Pic}^1_T \to \mathrm{Pic}^1_T$. We consider the line bundle

$$\mathcal{N} = \Big(\det R(p_2)_*\big(p_1^*(\mathcal{E}) \otimes \mathcal{L}\big)\Big)^{\otimes -1}$$

on Pic^1_T. This line bundle has a natural section (the theta function) $\theta(\mathcal{E})$, which extends the section 1 on the open schematically dense subscheme $U \subset \mathrm{Pic}^1_T$ over which the complex $R(p_2)_*\big(p_1^*(\mathcal{E}) \otimes \mathcal{L}\big)$ is trivial. We define $\mathcal{D}_{\mathcal{E}}$ to be the zero set of the section $\theta(\mathcal{E})$; it is also the largest closed subscheme of Pic^1_T over which the coherent $\mathcal{O}_{\mathrm{Pic}^1_T}$-module $R^1(p_2)_*\big(p_1^*(\mathcal{E}) \otimes \mathcal{L}\big)$ has rank ≥ 1 (i.e., defined in terms of a fitting ideal of this sheaf).

One can prove that $\mathcal{N} \otimes \mathcal{O}(-2\Theta^1_T)$ is isomorphic to the pullback of an invertible $\mathcal{O}_T$-module on T. Hence the divisor $\mathcal{D}_{\mathcal{E}}$ defines an S-morphism of T into $\mathbf{P}$:

$$\varphi_{(T,\mathcal{E})} : T \longrightarrow \mathbf{P}.$$

These constructions define therefore a transformation of the stack of semi-stable rank 2 bundles with trivial determinant on $\mathcal{C}$ over S towards the scheme $\mathbf{P}$. It turns out that this defines an isomorphism of the coarse moduli scheme

$$\mathcal{M}_{\mathcal{C}} \longrightarrow \mathbf{P}$$

towards $\mathbf{P}$, see [NR].

A remark about the natural determinant bundle $\mathcal{L}$ on $\mathcal{M}_{\mathcal{C}}$ (see [Hi, page 360]). The relative Picard group of $\mathbf{P}$ over S is $\mathbf{Z}$ and is generated by $\mathcal{O}_{\mathbf{P}}(1)$. Hence it is clear that $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}}(n) \otimes p^*(\mathcal{N})$ for some invertible $\mathcal{O}_S$-module $\mathcal{N}$ on S. The integer n may be determined as follows. We know by [Hi, page 360] that $\mathcal{O}_P(n)^{-\lambda} \cong K_P$. The integer $\lambda = 4$ in this case, hence $n = 1$. We are going to define a projective heat operator on $\mathcal{L}^{\otimes k}$ over S, but we have seen in Subsection 2.3.5 that this is the same as defining a projective heat operator on $\mathcal{O}_{\mathbf{P}}(k)$. Hence we will work with the line bundle $\mathcal{O}_{\mathbf{P}}(k)$ from now on. (Note that as both $\mathcal{O}_{\mathbf{P}}(1)$ and $\mathcal{L}$ are defined on the moduli stack, they must be related by a line bundle coming from the moduli stack $\mathcal{M}_2$ of curves of genus 2; however $\mathrm{Pic}(\mathcal{M}_2)$ is rather small.)

Let $\mathrm{Pic}^0 = \mathrm{Pic}^0_{\mathcal{C}/S} \to S$ denote the Picard scheme of invertible $\mathcal{O}_{\mathcal{C}}$ bundles of relative degree 0 on $\mathcal{C}$ over S. There is a natural S-morphism

$$f : \mathrm{Pic}^0 \longrightarrow \mathbf{P}$$

which in terms of the moduli-interpretations of both spaces can be defined as follows:

$$\mathrm{Pic}^0(C) \ni [L] \longmapsto [L \oplus L^{-1}] \in \mathbf{P}(\mathbf{C}).$$

It is clear that the divisor D_E associated to $E = L \oplus L^{-1}$ is equal to $D_E = (\Theta^1_C + [L]) \cup (\Theta^1_C + [L^{-1}]) \subset \mathrm{Pic}^1(C)$. This implies readily that $f^*\mathcal{O}_{\mathbf{P}}(1) \cong \mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0)$, in fact $f^*\mathcal{O}_{\mathbf{P}}(1)$ may serve as the definition of $\mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0)$ on Pic^0. This induces an isomorpism $\mathbf{P} \cong |\mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0)|^*$ which identifies f with the natural map. Thus f factors over $\mathcal{K} := \mathrm{Pic}^0/\langle \pm 1 \rangle$, the relative Kummer surface, and defines a closed immersion (over S):

$$\mathcal{K} \hookrightarrow \mathbf{P}.$$

Any E on C that is semi-stable but not stable is an extension of the form $0 \to L \to E \to L^{-1} \to 0$, with L of degree zero. This bundle is S-equivalent to the bundle $L \oplus L^{-1}$. Thus the open subscheme $\mathbf{P}^s \subset \mathbf{P}$ is equal to the complement of the image of $f : \mathrm{Pic}^0 \to \mathbf{P}$.

2.5.2 Automorphisms

The group scheme $\mathcal{H} = \mathrm{Pic}^0[2]$ of 2-torsion points of Pic^0 over S acts on the schemes Pic^1, Indeed, if $\mathcal{E}$ over $\mathcal{C}_T$ is a family of locally free sheaves as above and if $\mathcal{A}$ is a line bundle on $\mathcal{C}_T$ which defines a 2-torsion point of Pic^0, then $\mathcal{E} \otimes \mathcal{A}$ is another family of locally free sheaves satisfying (a) and (b) of 2.5.1. This defines an action $\mathcal{H} \times_S \mathbf{P} \to \mathbf{P}$ of $\mathcal{H}$ on $\mathbf{P}$, given the identification of $\mathbf{P}$ as the moduli scheme. It is easy to see that this action is induced from the natural action of $\mathcal{H}$ on Pic^1. Furthermore the morphism $f : \mathrm{Pic}^0 \to \mathbf{P}$ is equivariant with respect to this action.

Note that the action just defined does not lift to an action of $\mathcal{H}$ on $\mathcal{O}_{\mathbf{P}}(1)$. Let us write $\mathcal{G}$ for theta group of the relatively ample line bundle $\mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0)$ on Pic^0 over S; this group scheme gives the Heisenberg group for any point $0 \in S(\mathbf{C})$. We remark that $\mathcal{G}$ fits into the exact sequence

$$1 \longrightarrow \mathbf{G}_{m,S} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

There is a unique action of $\mathcal{G}$ on $\mathcal{O}_{\mathbf{P}}(1)$ which lifts the action of $\mathcal{H}$ on $\mathbf{P}$ and agrees with the defining action $\mathcal{G}$ on $f^*\mathcal{O}_{\mathbf{P}}(1)$ over Pic^0.

2.5.3 Test families

Let $\mathcal{E}$ be a rank 2 free $\mathcal{O}$-module on $\mathcal{C}_T$ satisfying (a) and (b) of 2.5.1. We will say that the pair $(T, \mathcal{E})$ is a *test family* if the following conditions are satisfied:

1. $T \to S$ is smooth,
2. all $E = \mathcal{E}_0$ are simple vector bundles on $C = \mathcal{C}_0$, and
3. the map $\mathcal{T}_{T/S} \longrightarrow R^1\mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}))$ is an isomorphism.

The last condition needs some clarification. The obstruction for the locally free sheaf $\mathcal{E}$ to have a connection on $\mathcal{C}_T$ is an element in

$$H^1(\mathcal{C} \times_S T, \Omega^1_{\mathcal{C}\times_S T} \otimes \mathcal{E}nd(\mathcal{E})).$$

We can use the maps $\Omega^1_{\mathcal{C}\times_S T} \to \mathrm{pr}_2^*\Omega^1_{T/S}$ and $\mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd_0(\mathcal{E})$ to project this to a section of $\Omega^1_{T/S} \otimes R^1\mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}))$. Whence the map of condition 3. This condition means that the deformation of $E = \mathcal{E}_0$ for any $0 \in T$ is versal in the "vertical direction".

The following lemma is proved in the usual manner, using deformation theory (no obstructions !) and Artin approximation.

2.5.4 Lemma. For any $0 \in S(\mathbf{C})$ and any simple bundle E on $C = \mathcal{C}_0$ there exists a test family $(T, \mathcal{E})$ such that (E, C) occurs as one of its fibres.

2.5.5 Lemma. Let $(T, \mathcal{E})$ be a test family. The morphism $\varphi_{(T,\mathcal{E})} : T \to \mathbf{P}$ is étale.

Proof. Let E be a simple rank 2 bundle on a smooth projective genus two curve C over $\mathrm{Spec}(\mathbf{C})$ with $\det(E) = \mathcal{O}_C$. Let $\eta \in H^1(C, \mathcal{E}nd_0(E))$. For each invertible sheaf L on C, with $\deg(L) = -1$ and given embedding $L \subset E$ consider the map

$$\mathcal{E}nd_0(E) \longrightarrow \mathcal{H}om(L, E).$$

We have to show $(*)$: If for all $L \subset E$ as above η maps to 0 in $H^1(C, \mathcal{H}om(L, E))$, then $\eta = 0$.

Indeed, the condition $\eta \mapsto 0$ means that the trivial deformation $L[\epsilon]$ of L over $C[\epsilon] = C \times \mathrm{Spec}(\mathbf{C}[\epsilon])$ can be embedded into the deformation of E given by η. If this holds for all

$L \subset E$, then the divisor D_E does not move, and we want this to imply that the infinitesimal deformation of E is trivial.

Let us prove $(*)$. We will do this in the case that E is not stable, as we already have the result in the stable case, see Subsection 2.4.1. Thus we may assume the bundle E is simple but not stable: E is a nontrivial extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow A^{-1} \longrightarrow 0,$$

with $\deg_C A = 0$, $A^{\otimes 2} \not\cong \mathcal{O}_C$. In this case it is easy to see that there exist three line bundles $L_1 \subset E$, $L_2 \subset E$ and $L_3 \subset E$ of degree -1 on C such that the arrow towards E in the following exact sequence is surjective:

$$0 \longrightarrow K \longrightarrow L_1 \oplus L_2 \oplus L_3 \longrightarrow E \longrightarrow 0.$$

Here K is just the kernel of the surjection. For example we can take L_i of the form $A^{-1}(-P_i)$ for $i = 1, 2$ and L_3 of the form $A(-P_3)$. (This is the only thing we will need in the rest of the argument; it should be easy to establish this in the case of a stable bundle E also.) This property then holds for $L_i \subset E$ sufficiently general also; choose L_i sufficiently general. Note that $K \cong L_1 \otimes L_2 \otimes L_3$ as $\det(E) = \mathcal{O}_C$.

Suppose we have a deformation E_ϵ of E given by η as above, i.e., we have $L_i[\epsilon] \to E_\epsilon$ lifting $L \to E$. This determines an exact sequence

$$0 \longrightarrow K_\epsilon \longrightarrow L_1[\epsilon] \oplus L_2[\epsilon] \oplus L_3[\epsilon] \longrightarrow E_\epsilon \longrightarrow 0.$$

But $\det E_\epsilon \cong \mathcal{O}_\mathcal{C}[\epsilon]$, hence $K_\epsilon \cong L_1[\epsilon] \otimes L_2[\epsilon] \otimes L_3[\epsilon] \cong (L_1 \otimes L_2 \otimes L_3)[\epsilon]$ is trivial as well. Now note that by our general position we have

$$\dim H^0(C, \mathcal{H}om(K, L_1)) = \dim H^0(C, L_2 \otimes L_3) = 1. \tag{1}$$

Similar for L_2 and L_3. Therefore, up to a unit in $\mathbf{C}[\epsilon]$, there is only one map $K[\epsilon] \to L_i[\epsilon]$, lifting $K \to L_i$. Thus the sequence (1) above is uniquely determined up to isomorphism, hence $E_\epsilon \cong E[\epsilon]$ is constant, i.e., $\eta = 0$. QED

2.6 Results

This lemma completes the preparations. In the remainder of this section we prove the desired results on the existence and the characterization of the connection.

2.6.1 Theorem. The Hitchin symbol $\rho^s_{Hitchin} : p^*\mathcal{T}_S \to S^2\mathcal{T}_{\mathbf{P}^s/S}$ defined in Subsection 2.4.1 extends to a symbol

$$\rho_{Hitchin} : p^*\mathcal{T}_S \longrightarrow S^2\mathcal{T}_{\mathbf{P}/S}.$$

Proof. Let $(T, \mathcal{E})$ be a test family. We have the morphism $\varphi = \varphi_{(T,\mathcal{E})} : T \to \mathbf{P}$ which is étale and wich induces therefore an isomorphism $\mathcal{T}_{T/S} \to \varphi^*\mathcal{T}_{\mathbf{P}/S}$. On the other hand we have the isomorphism $\mathcal{T}_{T/S} \to R^1\mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}))$ of the test family $(T, \mathcal{E})$. We also have the pairing as in Subsection 2.4.1

$$\mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}) \otimes \Omega^1_{\mathcal{C}_T/T}) \otimes \mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}) \otimes \Omega^1_{\mathcal{C}_T/T}) \longrightarrow \mathrm{pr}_{2*}((\Omega^1_{\mathcal{C}_T/T})^{\otimes 2}).$$

Note that $\mathrm{pr}_{2*}(\mathcal{E}nd_0(\mathcal{E}) \otimes \Omega^1_{\mathcal{C}_T/T})$ is a locally free sheaf of $\mathcal{O}_T$-modules as $\mathcal{E}$ has simple fibres. These maps and duality give us together with the Kodaira-Spencer map a map

$$\mathcal{T}_S \otimes \mathcal{O}_T \longrightarrow S^2\varphi^*\mathcal{T}_{\mathbf{P}/S}.$$

We remark that on the (nonempty) stable locus $T^s = \varphi^{-1}(\mathbf{P}^s)$ this map is equal to the pullback of $\rho^s_{Hitchin}$.

Thus, by Lemma 2.5.5, we see that we can extend $\rho^s_{Hitchin}$ to any point $x \in \mathbf{P}(\mathbf{C})$ which is the image of some simple bundle. Note that a semi-stable bundle E, which is a nontrivial extension $0 \to L \to E \to L^{-1} \to 0$ with $deg(L) = 0$ is simple if and only if $L^{\otimes 2} \not\cong \mathcal{O}_C$. It follows from Lemma 2.5.4 and the above that we can extend $\rho^s_{Hitchin}$ to the complement of the codimension 3 locus $f(\mathrm{Pic}^0[2])$, and hence by Hartog's theorem it extends. QED

2.6.2 Proposition. Let $f : \mathrm{Pic}^0 \to \mathbf{P}$ be as in 2.5.1, then f is a submersion on the locus $\mathcal{Y} := \mathrm{Pic}^0 \setminus \mathrm{Pic}^0[2]$. There is a commutative diagram

$$\begin{array}{ccccc} f^*(\rho_{Hitchin}) & : & f^*p^*\mathcal{T}_S & \longrightarrow & f^*S^2\mathcal{T}_{\mathbf{P}/S} \\ & & \| & & \uparrow \\ 4\rho_{Ab}|_{\mathcal{Y}} & : & \mathcal{T}_S \otimes \mathcal{O}_{\mathcal{Y}} & \longrightarrow & S^2\mathcal{T}_{\mathcal{Y}/S} \end{array}$$

where ρ_{Ab} is the symbol of the abelian heat operator on $\mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0) = f^*\mathcal{O}_{\mathbf{P}}(1)$.

Proof. Consider once again a simple bundle E, which is given as an extension $0 \to L \to E \to L^{-1} \to 0$ with $deg(L) = 0$. Clearly, the kernel of the map

$$H^1(C, \mathcal{E}nd_0(E)) \longrightarrow H^1(C, \mathcal{H}om(L, L^{-1}))$$

represents deformations preserving the filtration on E, i.e., tangent vectors along the Kummer surface. Dually, this corresponds to the image of the map

$$H^0(C, \mathcal{H}om(L^{-1}, L) \otimes \Omega^1_C) \longrightarrow H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1_C).$$

We use the Trace-form to identify $\mathcal{E}nd_0(E)$ with its dual. Note that there is a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}nd_0(E)/L^{\otimes 2} \longrightarrow L^{\otimes -2} \longrightarrow 0$$

induced by the filtration $L \subset E$ on E. Note that $\dim H^0(C, (\mathcal{E}nd_0(E)/L^{\otimes 2}) \otimes \Omega^1_C) = 2$, since if it were ≥ 3, then $\dim H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1) \geq 4$ contrary to the assumption that E is simple. Thus we get the exact sequence

$$H^0(C, \Omega^1) \cong H^0(C, (\mathcal{E}nd_0(E)/L^{\otimes 2}) \otimes \Omega^1_C) \twoheadleftarrow H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1_C) \hookleftarrow H^0(C, L^{\otimes 2} \otimes \Omega^1_C) \cong \mathbf{C}.$$

This is (canonically) dual to the sequence of cotangent spaces

$$0 \longrightarrow T_{[L]}\mathrm{Pic}^0 \longrightarrow T_{f([L])}(\mathbf{P}) \longrightarrow (N_{\mathcal{Y}}\mathbf{P})_{[L]} \longrightarrow 0.$$

We want to see that our symbols in the point $[L]$ lie in the space $S^2T_{[L]}\mathrm{Pic}^0$.

We have to study the map

$$\begin{array}{ccc} H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1_C) \times H^0(C, \mathcal{E}nd_0(E) \otimes \Omega^1_C) & \longrightarrow & H^0(C, (\Omega^1_C)^{\otimes 2}) \\ X \otimes \eta \times Y \otimes \omega & \longmapsto & \mathrm{Trace}(XY) \otimes \eta\omega. \end{array}$$

But the elements of $H^0(C, L^{\otimes 2} \otimes \Omega^1_C)$, resp. those of $H^0(C, \Omega^1_C)$ locally look like

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \otimes \eta, \qquad \text{resp.} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \omega.$$

Therefore the pairing just reduces to twice the multiplication pairing $H^0(C, \Omega^1_C) \times H^0(C, \Omega^1_C) \to H^0(C, (\Omega^1_C)^{\otimes 2})$.

This multiplication pairing, however, corresponds excactly (via Kodaira-Spencer) to the symbol of the heat operator on the theta-divisor of the Jacobian Pic^0 of $\mathcal{C}$ over S. Since we are looking at $2\Theta^0$, we get the desired factor 4, compare with Subsections 2.3.8 and 2.3.7. QED

2.6.3 Corollary. The symbol $\rho_{Hitchin}$ is invariant under the action of the group scheme $\mathcal{H} = \mathrm{Pic}^0[2]$ on $\mathbf{P}$ and $S^2\mathcal{T}_{\mathbf{P}/S}$.

Proof. This follows from the observation that it is true for $\rho_{Ab|\mathcal{Y}}$ and thus for $\rho_{Hitchin}|_{\mathcal{K}}$, the Hitchin symbol restricted to the Kummer surface. Further, one uses the remark that there are no nonvanishing elements of $H^0(\mathbf{P}, S^2\mathcal{T}_{\mathbf{P}/S})$ which vanish on $\mathcal{K}$, that is $H^0(\mathbf{P}, S^2\mathcal{T}_{\mathbf{P}/S}(-4)) = 0$. QED

2.6.4 Proposition. There exists a unique projective heat operator $\bar{D}_{\lambda,k}$ (with $\lambda \in \mathbf{C}^*$ be a nonzero complex number) on $\mathcal{O}_{\mathbf{P}}(k)$ over S with the following properties:

1. the symbol of $\bar{D}_{\lambda,k}$ is equal to $\frac{1}{4\lambda}\rho_{Hitchin}$, and
2. the operator $\bar{D}_{\lambda,k}$ commutes (projectively) with the action of $\mathcal{G}$ on $\mathcal{O}_{\mathbf{P}}(k)$.

Proof. This follows readily from the discussion in Subsection 2.3.7. Indeed, both obstructions mentioned there vanish in view of the vanishing of $R^1p_*\mathcal{T}_{\mathbf{P}/S}$ and $R^1p_*\mathcal{O}_{\mathbf{P}}$. We have a good way of choosing the elements $D(\theta)$: namely, we choose them $\mathcal{G}$-invariantly. This is possible: first choose arbitrary lifts, and then average over liftings of a full set of sections of $\mathcal{H}$ (this can be done étale locally over S). This determines the $D(\theta)$ uniquely up to an element of $p^{-1}(\mathcal{O}_S)$, as $(p_*\mathcal{T}_{\mathbf{P}/S})^{\mathcal{H}} = (0)$. Hence we get the desired projective heat operator. QED

It follows from Proposition 2.6.2 that the heat operators $\bar{D}_{\lambda,k}$ are compatible with the morphism $f : \mathcal{Y} \to \mathbf{P}$. Also, for $k = 1$ and $\lambda = 1$, we see that $\bar{D}_{1,1}$ is compatible with the abelian heat operator on $\mathcal{O}_{\mathrm{Pic}^0}(2\Theta^0)$ restricted to $\mathcal{Y}$ (this is one of the reasons for the factor $\frac{1}{4}$, but see below also).

2.6.5 Hitchin's connection for genus 2 and rank 2

The title of this subsection is somewhat misleading. As mentioned before, in the paper [Hi] there is no definition of a heat operator in the case of genus 2 and rank 2. However, we propose the following definition.

2.6.6 Definition.

The (projective) Hitchin connection on $p_*\mathcal{O}_{\mathbf{P}}(k)$ is the projective connection associated to the projective heat operator $\mathbf{D}_k := \bar{D}_{(k+2),k}$.

This definition makes sense for the following reason: The symbol of the operator $\bar{D}_{(k+2),k}$ is equal to $2/(2k+4)\rho_{Hitchin}$ which is equal to the symbol that occurs in [Hi, Theorem 3.6] (the factor 2 comes from our way of normalizing symbols, see Subsection 2.2.4). Thus the only assumption we needed in order to get $\mathbf{D}_k$ was the assumption that it is compatible with the action of $\mathcal{G}$, see Subsection 2.6.4.

2.6.7 How to compute the heat operators?

To determine $\mathbf{D}_k$ we introduce heat operators $D_{\lambda,k}$ under the following assumptions:

1. The family of Kummer surfaces $\mathcal{K} \subset \mathbf{P}$ is given by one equation $F \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(4))$.
2. We have chosen an integrable connection ∇_0 on $\mathcal{F} := p_*\mathcal{O}_{\mathbf{P}}(1)$ over S which is equivariant for the action of $\mathcal{G}$.

We remark that both conditions can be satisfied on the members of an open covering of S. Also the connection ∇_0 in 2 is determined uniquely up to the addition of a linear operator of the form $\eta \cdot \mathrm{id}$ for some closed 1-form η on S. We remark that ∇_0 induces a connection ∇_0 on all the locally free sheaves $S^k\mathcal{F} = p_*\mathcal{O}_{\mathbf{P}}(k)$ over S, but these should not be confused with the connections induced by the $D_{\lambda,k}$! In addition ∇_0 determines an integrable connection on $\mathbf{P}$ over S and a lift of this to connections on the sheaves $\mathcal{O}_{\mathbf{P}}(k)$ over $\mathbf{P}$. In other words we have rigidified $(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k))$ over S.

There is a natural surjection

$$S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^* \longrightarrow p_*S^2\mathcal{T}_{\mathbf{P}/S}.$$

which has a canonical splitting, given by decomposing $S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^*$ into irreducible $\mathrm{GL}(\mathcal{F})$-modules, i.e., the unique $\mathrm{GL}(\mathcal{F})$-equivariant splitting. Thus we identify a section X of $S^2\mathcal{T}_{\mathbf{P}/S}$ with a section X of $S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^*$. Note that we can regard sections of $S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^*$ as sections of $\mathrm{Diff}^{(2)}_{\mathbf{P}/S}(\mathcal{O}_{\mathbf{P}}(k))$ in a natural manner, by considering them as second order differential operators on the affine 4-space $\underline{\mathrm{Spec}}(S^*\mathcal{F})$ invariant under the scalar action of $\mathbf{G}_{m,S}$. We write E for the Euler-vector field, i.e., the section of $\mathcal{F} \otimes \mathcal{F}^*$ that corresponds to the identity map of $\mathcal{F}$.

Assume we have a $\mathcal{G}$-invariant section

$$X \in S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^* \otimes_{\mathcal{O}_S} \Omega^1_S$$

with the following property

$$(Eq.) \qquad X \cdot F = \nabla_0(F)E \quad \text{in} \quad S^5(\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{F}^* \otimes_{\mathcal{O}_S} \Omega^1_S,$$

where $X \cdot F$ denotes contracting once.

If we have such an X we define a heat operator

$$D_{\lambda,k} : p^*\mathcal{T}_S \longrightarrow \mathrm{Diff}^{(2)}_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}(k))$$

by the formula (θ a local section of $\mathcal{T}_S$)

$$D_{\lambda,k}(\theta) = \theta - \frac{1}{\lambda} X_{\theta,k}.$$

Here $X_{\theta,k}$ means the following: first contract X with θ to get a section X_θ of $S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^*$ and then consider this as a second order differential operator $X_{\theta,k}$ of $\mathcal{O}_{\mathbf{P}}(k)$ on X over S by the remarks above. Note that the term θ acts on $\mathcal{O}_{\mathbf{P}}(k)$ using ∇_0.

2.6.8 Lemma. Let $D_{\lambda,k}$ be the heat operator defined above.

1. The heat operator $D_{\lambda,k}$ commutes with the action of $\mathcal{G}$ on $\mathcal{O}_{\mathbf{P}}(k)$.
2. The heat operators $D_{\lambda,k}$ are compatible with $f : \mathcal{Y} \to \mathbf{P}$. If $\lambda = k$ then $D_{k,k}$ is strictly compatible with $f : \mathcal{Y} \to \mathbf{P}$.

Proof. The second statement holds since X is $\mathcal{G}$ invariant. We now verify the last statement. Note that $D_{k,k}$ is strictly compatible with $\mathcal{K} \to \mathbf{P}$ if the operators $D_{k,k}(\theta)$ preserve the subsheaf $\mathcal{I}_{\mathcal{K}} \cdot \mathcal{O}_{\mathbf{P}}(k)$ of $\mathcal{O}_{\mathbf{P}}(k)$. This can be seen by the description of strict compatiblility in terms of local coordinates given in Subsection 2.3.10. We may check this condition on the affine 4-space $\underline{\mathrm{Spec}}(S^*\mathcal{F})$. Thus let $G \in S^{k-4}(\mathcal{F})$ be a section, let θ be a local vector field on S and consider

$$\nabla_{0,\theta}(GF) - \frac{1}{k} X_\theta(GF)$$

We have to show that this is divisible by F.

We may assume that G is horizontal for ∇_0, as we can find a horizontal basis for $S^{k-4}(\mathcal{F})$ locally. Thus we get

$$G\nabla_{0,\theta}(F) - \frac{1}{k} X_\theta(GF) = G\nabla_{0,\theta}(F) - \frac{1}{k} X_\theta(G)F - \frac{1}{k}(X \cdot F)_\theta(G) - \frac{1}{k} G X_\theta(F).$$

Here we have used a general formula for the application of a second order operator like X_θ on a product like FG. Note also that $X(F)$ is simply the contration of $X \cdot F$, hence by $(Eq.)$ we get

$$G\nabla_{0,\theta}(F) - \frac{1}{k} X_\theta(G)F - \frac{1}{k}\nabla_{0,\theta}(F)E(G) - \frac{1}{k} G E(\nabla_{0,\theta}(F)).$$

Next, we use that acting by E on a homogeneous polynomial gives degree times the polynomial:

$$G\nabla_{0,\theta}(F) - \frac{1}{k} X_\theta(G)F - \frac{1}{k}\nabla_{0,\theta}(F)(k-4)G - \frac{1}{k} G4\nabla_{0,\theta}(F) = -\frac{1}{k} X_\theta(G)F$$

which is divisible by F. Thus $D_{k,k}$ is indeed stricly compatible with $f : \mathcal{Y} \to \mathbf{P}$. QED

2.6.9 Theorem. Let X be a solution to the equation $(Eq.)$ above and let $D_{\lambda,k}$ be the heat operators defined in this subsection. Then

1. The projective heat operator defined by $D_{\lambda,k}$ is the operator $\bar{D}_{\lambda,k}$ defined in Subsection 2.6.4. Thus $D_{k+2,k}$ defines the Hitchin connection on $\mathcal{O}_{\mathbf{P}}(k)$.
2. The heat operators $D_{\lambda,k}$ are projectively flat for any k and λ.
3. For $\lambda = k$, the operator $D_{k,k}$ is strictly compatible with the abelian projective heat operator on $\mathcal{O}_{\mathrm{Pic}^0}(\mathcal{O}(2k\Theta^0))$ over an open part of Pic^0.

Proof. Consider first the case of $D_{k,k}$ as defined in 2.6.7. We have seen above that it is strictly compatible with $f : \mathcal{Y} \to \mathbf{P}$. Therefore, by Subsection 2.3.10 it defines a heat operator D_k on $f^*\mathcal{O}_{\mathbf{P}}(k)$ over $\mathcal{Y}$. In view of Hartog's theorem this extends uniquely to a heat operator D_k of $\mathcal{O}_{\mathrm{Pic}^0}(\mathcal{O}(2k\Theta^0))$ on Pic^0 over S. We have seen in Subsection 2.3.8 that there is a unique such heat operator, whose symbol is $1/k$ times the symbol ρ_{Ab} of D_1. Combining this with the assertions in Subsection 2.6.2 we see that $(1/4k)\rho_{Hitchin} - \rho_{D_{k,k}}$ vanishes along $\mathcal{K}$ and hence is zero. This proves that $D_{k,k}$ agrees with the projective heat operator $\bar{D}_{k,k}$ defined in Subsection 2.6.4.

Now it follows from the transformation behaviour of the symbol of $\bar{D}_{\lambda,k}$ and $D_{\lambda,k}$ that these agree for arbitrary λ. Thus we get the agreement stated in the theorem.

To see that these heat operators are projectively flat, we argue as follows. Let θ, θ' be two local commuting vector fields on S. We have to see that the section $[D_{\lambda,k}(\theta), D_{\lambda,k}(\theta')]$ of $\mathrm{Diff}^{(3)}_{\mathbf{P}/S}(\mathcal{O}_{\mathbf{P}}(k))$ lies in the subsheaf $p^{-1}(\mathcal{O}_S)$. Again, for $\lambda = k$ this section is zero when restricted to $\mathcal{K}$, and again this implies that the section is zero in that case. Now, let us compute

$$[D_{\lambda,k}(\theta), D_{\lambda,k}(\theta')] = [\theta - \frac{1}{\lambda}X_{\theta,k}, \theta' - \frac{1}{\lambda}X_{\theta',k}] = \frac{1}{\lambda}\Big([\theta, X_{\theta',k}] - [\theta', X_{\theta,k}]\Big) + \frac{1}{\lambda^2}[X_{\theta,k}, X_{\theta',k}].$$

The index k is now superfuous, as we can see this as an expression in $S^2\mathcal{F} \otimes_{\mathcal{O}_S} S^2\mathcal{F}^* \oplus S^3\mathcal{F} \otimes_{\mathcal{O}_S} S^3\mathcal{F}^*$. We know that this expression is zero for any k with $\lambda = k$ considered as a third order operator on $\mathcal{O}_{\mathbf{P}}(k)$. This implies that both terms are zero, hence the expression is zero for any λ and any k. (We remark for the doubtful reader that we will verify the flatness also by a direct computation using the explicit description of the operator.) QED

3 The flat connection

3.1

We introduce certain families of flat connections on (trivial) bundles over the configuration space $\mathcal{P}$ (cf. 1.4). These are defined by representations of the Lie algebra $so(2g+2)$. Then we derive a convenient form for the equation of the family of Kummer surfaces $\mathcal{K} \hookrightarrow \mathbf{P}$ (in fact for any g we point out a specific section P_z of a trivial bundle over $\mathcal{P}$). In Theorem 3.4.3 we show that this equation is flat for one of our connections. This will be important in identifying Hitchin's connection in the next section.

3.2 Orthogonal groups

3.2.1 The Lie algebra

Let $Q = x_1^2 + \ldots + x_{2g+2}^2$, then the (complex) Lie algebra $so(Q) = so(2g+2)$ is:

$$so(2g+2) = \{A \in End(\mathbf{C}^{2g+2}) : \ {}^tA + A = 0\ \}, \qquad \text{let} \quad F_{ij} := 2(E_{ij} - E_{ji})\ (\in\ so(2g+2)),$$

where E_{ij} is the matrix whose only non-zero entry is a 1 in the (i,j)-th position (the commutator of such matrices is $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$). The alternating matrices F_{ij} (with $1 \leq i < j \leq 2g+2$) are a basis of $so(2g+2)$. Note $F_{ij} = -F_{ji}$. These matrices satisfy the relations:

$$[F_{ij},\ F_{kl}] = \begin{cases} 0 & \text{if} \quad i,\ j,\ k,\ l \quad \text{are distinct,} \\ 2F_{il} & \text{if} \quad j = k. \end{cases}$$

3.2.2 The universal envelopping algebra

Recall that for a Lie algebra g the tensor algebra $T(g)$ and the universal enveloping algebra $U(g)$ are defined by:

$$T(g) := \mathbf{C} \oplus g \oplus g \otimes g \ldots, \qquad U(g) := T(g)/I$$

where I is the ideal generated by all elements of the form $x \otimes y - y \otimes x - [x,y]$ with $x,\ y \in g$. Lie algebra representations $\rho : g \to End(V)$ (so $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$ correspond to representations $\tilde{\rho} : U(g) \to End(V)$ of associative algebras with $\rho(1) = id_V$. Given ρ one defines $\tilde{\rho}(x_{i_1} \otimes x_{i_2} \ldots \otimes x_{i_k}) := \rho(x_{i_1})\rho(x_{i_2})\ldots\rho(x_{i_k})$ which is well-defined because we work modulo I.

3.2.3 Definitions

We define elements in $U(so(2g+2))$ by:

$$\Omega_{ij} := F_{ij} \otimes F_{ij} \mod I \qquad (\in U(so(2g+2))), \qquad \text{note} \quad \Omega_{ij} = \Omega_{ji}.$$

In particular, for any Lie algebra representation $\rho : so(2g+2) \to W$ we now have endomorphisms $\tilde{\rho}(\Omega_{ij}) = \rho(F_{ij})^2 : W \to W$ which satisfy the same commutation relations as the Ω_{ij}.

For $\lambda \in \mathbf{C}^*$ we define an $U(so(2g+2))$-valued one form ω_λ on $\mathcal{P}$:

$$\omega_\lambda := \lambda^{-1}\left(\sum_{i \neq j} \frac{\Omega_{ij}\mathrm{d}z_i}{z_i - z_j}\right); \qquad \text{let} \quad \tilde{\rho}(\omega_\lambda) := \lambda^{-1}\left(\sum_{i \neq j} \frac{\tilde{\rho}(\Omega_{ij})\mathrm{d}z_i}{z_i - z_j}\right)$$

where $\tilde{\rho} : U(so(2g+2)) \to End(W)$ is a representation. Note $\tilde{\rho}(\omega_\lambda) \in \mathrm{End}(W) \otimes \Omega^1_{\mathcal{P}}$.

The one form ω_λ defines a connection on the trivial bundle $W \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{P}}$ by:

$$\nabla^W_\lambda : W \otimes \mathcal{O}_{\mathcal{P}} \longrightarrow W \otimes \Omega^1_{\mathcal{P}}, \qquad fw \longmapsto w \otimes \mathrm{d}f - f\tilde{\rho}(\omega_\lambda)(w) \qquad (f \in \mathcal{O}_{\mathcal{P}},\ w \in W).$$

The covariant derivative of $f\otimes w$ with respect to the vector field $\partial/\partial z_i$ on $\mathcal{P}$ is the composition of ∇^W_λ with contraction:

$$\left(\nabla^W_\lambda(f\otimes w)\right)_{\partial/\partial z_i} = w\otimes\frac{\partial f}{\partial z_i} - \sum_{j,j\neq i}\tilde\rho(\Omega_{ij})(w)\otimes\frac{1}{z_i-z_j} \qquad (\in W\otimes\mathcal{O}_{\mathcal{P}}).$$

3.2.4 Proposition. Let $\rho : so(2g+2) \longrightarrow End(W)$ be a Lie algebra representation, then ∇^W_λ is a flat connection on $W\otimes\mathcal{O}_{\mathcal{P}}$ for any $\lambda\in\mathbf{C}^*$.

Proof. Obviously ∇^W_λ is a connection. It is well known (and easy to verify) that the connection defined by ω is flat (so $\mathrm{d}\omega+\omega\wedge\omega=0$) if the following *infinitesimal pure braid relations* in $U(so(2g+2))$ are satisfied:

$$[\Omega_{ij},\Omega_{kl}]=0, \qquad [\Omega_{ik},\Omega_{ij}+\Omega_{jk}]=0,$$

where i, j, k, l are distinct indices (cf. [Ka], Section XIX.2).

To check these relations we use that $X\otimes Y = Y\otimes X + [X,Y]$ in $U(so)$. The first relation is then obviously satisfied. We spell out the second. Consider first:

$$\begin{aligned} F_{ik}\otimes F_{ij}\otimes F_{ij} &= (F_{ij}\otimes F_{ik}+[F_{ik},F_{ij}])\otimes F_{ij}\\ &= F_{ij}\otimes F_{ik}\otimes F_{ij} - F_{kj}\otimes F_{ij}\\ &= F_{ij}\otimes(F_{ij}\otimes F_{ik}-F_{kj}) + F_{jk}\otimes F_{ij}\\ &= F_{ij}\otimes F_{ij}\otimes F_{ik} + (F_{ij}\otimes F_{jk}+F_{jk}\otimes F_{ij}),\end{aligned}$$

so we have $[F_{ik},F_{ij}\otimes F_{ij}]=D_1$ with D_1 symmetric in the indices i and k. Similarly we get:

$$\begin{aligned} F_{ik}\otimes(F_{ik}\otimes F_{ij}\otimes F_{ij}) &= F_{ik}\otimes(F_{ij}\otimes F_{ij}\otimes F_{ik}+D_1)\\ &= (F_{ik}\otimes F_{ij}\otimes F_{ij})\otimes F_{ik} + F_{ik}\otimes D_1\\ &= F_{ij}\otimes F_{ij}\otimes F_{ik}\otimes F_{ik} + (D_1\otimes F_{ik}+F_{ik}\otimes D_1).\end{aligned}$$

Thus $[\Omega_{ik},\Omega_{ij}] = D_1\otimes F_{ik}+F_{ik}\otimes D_1$ which is antisymmeric in i and k. Therefore

$$[\Omega_{ik},\Omega_{ij}] = -[\Omega_{ki},\Omega_{kj}] = -[\Omega_{ik},\Omega_{jk}]$$

which proves the second infinitesimal braid relation. QED

3.2.5 Differential operators

Given a Lie algebra representation $\rho : g\to \mathrm{End}(\mathrm{S}_1)$ where $S_k\subset S := \mathbf{C}[\ldots,X_i,\ldots]$ is the subspace of homogeneous polynomials of degree k, there is a convenient way to determine the representation $\rho^{(k)} : g \to \mathrm{End}(S_k)$ induced by ρ. If $\rho(A) = (a_{ij})$ w.r.t. the basis X_i of S_1, then we define

$$L_A := \sum_{i,j} a_{ij} X_i \partial_j \qquad (\in \mathrm{End}(S)), \qquad \text{with} \quad \partial_j(P) := \frac{\partial P}{\partial X_j}$$

for $P \in S$. Then obviously $\rho(A)(P) = L_A(P)$ if P is linear and the Leibnitz rule shows that $\rho^{(k)}(A) = L_A : S_k \to S_k$ for all k.

The composition (in $\mathrm{End}(S)$) of two operators is given by:

$$(X_i\partial_j) \circ (X_k\partial_l) = X_i X_k \partial_j \partial_l + \delta_{jk} X_i \partial_l, \qquad \text{thus} \quad L_A^2 = (1/2)\sigma(L_A^2) + L_{A^2},$$

with symbol $\sigma(L_A^2) = 2\sum_{i,j,k,l} a_{ij} a_{kl} X_i X_k \partial_j \partial_l$ (see our convention 2.2.4).

Assume that we have $A_{ij} \in g$ such that $\Omega_{ij} = A_{ij} \otimes A_{ij}$ satisfy the infinitesimal braid relations (cf. proof of the previous proposition) and that in the representation ρ we have $\rho(A_{ij})^2 = \mu I$ with $\mu \in \mathbf{C}$. Then we conclude that the operators

$$\sigma(L_{A_{ij}}^2) : S \longrightarrow S$$

also satisfy the infinitesimal braid relations.

3.3 The Kummer equation.

3.3.1 Introduction

We recall how one can obtain the equation of the Kummer variety for a genus two curve, and more generally, how a hyperelliptic curve $\mathcal{C}_z$ determines in a natural way a quartic polynomial $P_z \in S^4 V(\omega_{g+1})$ where $V(\omega_{g+1})$ is a half spin representation of $so(2g+2)$.

The polynomial P_z lies in the subrepresentation of $V(4\omega_{g+1})$ which also occurs in $S^2(\wedge^{g+1}\mathbf{C}^{2g+2})$ where $\mathbf{C}^{2g+2}$ is the standard representation of $so(2g+2)$. We will exploit this fact to verify that P_z is a flat section for one of the connections introduced in Subsection 3.2.

3.3.2 The orthogonal Grassmanian

We define two quadratic forms on $\mathbf{C}^{2g+2}$ by:

$$Q: \quad x_1^2 + \ldots + x_{2g+2}^2, \qquad Q_z: \quad z_1 x_1^2 + \ldots + z_{2g+2} x_{2g+2}^2,$$

and we use the same symbols to denote the corresponding quadrics in $\mathbf{P}^{2g+1}$. The quadric Q has two rulings (families of linear $\mathbf{P}^g$'s lying on it). Each of these is parametrized by the orthogonal Grassmanian (spinor variety) denoted by Gr_{SO}. The variety Gr_{SO} smooth, projective of dimension $\frac{1}{2}g(g+1)$.

Let $V(\omega_{g+1})$ be a half spin representation of $so(2g+2)$. There is an embedding

$$\phi : Gr_{SO} \longrightarrow \mathbf{P}V(\omega_{g+1})^*, \qquad \text{let} \quad \mathcal{O}_{Gr_{SO}}(1) := \phi^* \mathcal{O}_{\mathbf{P}V}(1).$$

in fact, $\phi(Gr_{SO})$ is the orbit of the highest weight vector. The map ϕ is equivariant for the action of the spin group $\widetilde{SO}$ (which acts through a half-spin representation on $V(\omega_{g+1})$). It induces isomorphisms:

$$H^0(Gr_{SO}, \mathcal{O}(n)) = V(n\omega_{g+1}).$$

In case $g = 2$, one has isomorphisms:

$$\phi : Gr_{SO} \xrightarrow{\cong} \mathbf{P}^3, \qquad S_k = V(k\omega_{g+1}) \qquad \text{and} \quad \widetilde{SO}_6 \cong SL_4(\mathbf{C})$$

and the halfspin representation is identified with the standard representation $SL_4(\mathbf{C})$ on $\mathbf{C}^4$ (or its dual).

3.3.3 The Plücker map

Let Gr_{SL} be the Grassmanian of g-dimensional subspaces of $\mathbf{P}^{2g+1}$, we denote the Plücker map by:

$$p : Gr_{SL} \longrightarrow \mathbf{P} \wedge^{g+1} \mathbf{C}^{2g+2}, \qquad \langle v_1, \ldots, v_{g+1} \rangle \longmapsto v_1 \wedge \ldots \wedge v_{g+1} = \sum p_{i_1 \ldots i_{g+1}} e_{i_1} \wedge \ldots \wedge e_{i_{g+1}}.$$

The map p is equivariant for the $SL(2g+2)$-action on both sides and induces isomorphisms of $sl(2g+2)$-representations

$$H^0(Gr_{SL}, \mathcal{O}_{Gr_{SL}}(n)) = V(n\lambda_{g+1}), \qquad \text{with} \quad V(\lambda_{g+1}) \cong \wedge^{g+1}\mathbf{C}^{2g+2}.$$

As a representation of $so(2g+2)$ the space $\wedge^{g+1}\mathbf{C}^{2g+2}$ is reducible, one has (cf. [vG] (3.7), [FH], §19.2, Th. 19.2 and Remarks (ii)), with $V(\omega_g$ and $V(\omega_{g+1})$ the half spin representations:

$$\wedge^{g+1}\mathbf{C}^{2g+2} = V(2\omega_g) \oplus V(2\omega_{g+1}).$$

In particular, the irreducible component $V(4\omega_{g+1})$ of $S^4V(\omega_{g+1})$ is also a component of $S^2(\wedge^{g+1}\mathbf{C}^{2g+2})$ (viewed as $so(2g+2)$ representation). Elements of $S^2(\wedge^{g+1}\mathbf{C}^{2g+2})$ can be seen as quadratic forms in the Plücker coordinates, when restricted to Gr_{SO} they can be viewed as (restrictions to Gr_{SO} of) quartic polynomials on $\mathbf{P}V(\omega_{g+1})^*$.

3.3.4 Notation

The following lemma gives this decomposition of $\wedge^{g+1}\mathbf{C}^{2g+2}$ explicitly. Let $\{e_i\}$ be the standard basis of $\mathbf{C}^{2g+2}$. For any

$$S \subset B := \{1, 2, \ldots, 2g+2\}, \qquad |S| = g+1,$$

we write $S = \{i_1, i_2, \ldots, i_{g+1}\}$ with $i_1 < i_2 < \ldots < i_{g+1}$ and define

$$e_S := e_{i_1} \wedge \ldots \wedge e_{i_{g+1}} \qquad \in \wedge^{g+1}\mathbf{C}^{2g+2}.$$

These elements give a basis of $\wedge^{g+1}\mathbf{C}^{2g+2}$. For such an S we let $S' := B - S$ be the complement of S in B. Writing $S' = \{j_1, j_2, \ldots, j_{g+1}\}$ with $j_1 < j_2 < \ldots < j_{g+1}$, we define an element σ_S in the symmetric group S_{2g+2} by:

$$\sigma_S(k) := i_k, \quad \sigma_S(g+1+k) := j_k.$$

3.3.5 Lemma. The two non-trivial $so(2g+2)$-invariant subspaces in $\wedge^{g+1}\mathbf{C}^{2g+2}$ are:

$$\langle \ldots, e_S + sgn(\sigma_S) i^{g+1} e'_S, \ldots \rangle_{S\ni 1} \qquad \text{and} \qquad \langle \ldots, e_S - sgn(\sigma_S) i^{g+1} e'_S, \ldots \rangle_{S\ni 1},$$

where S runs over the subsets of B with $g+1$ elements with $1 \in S$.

Proof. (Cf. [FH], §19.2 Remarks(iii).) The quadratic form Q defines an so-equivariant isomorphism

$$B: \wedge^{g+1}\mathbf{C}^{2g+2} \longrightarrow (\wedge^{g+1}\mathbf{C}^{2g+2})^* = \wedge^{g+1}(\mathbf{C}^{2g+2})^*, \qquad e_S \longmapsto \epsilon_S := \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_{g+1}}$$

with $\{\ldots, \epsilon_j, \ldots\}$ the basis dual to the e_i. There is also a canonical (in particular so-equivariant) isomorphism:

$$C: \wedge^{g+1}\mathbf{C}^{2g+2} \longrightarrow (\wedge^{g+1}\mathbf{C}^{2g+2})^*, \qquad \alpha \longmapsto [\beta \mapsto c_{\alpha,\beta}]$$

with $c_{\alpha,\beta} \in \mathbf{C}$ defined by

$$\alpha \wedge \beta = c_{\alpha,\beta} e_1 \wedge e_2 \wedge \ldots e_{2g+2}.$$

Thus we have an isomorphism $A := B^{-1} \circ C: \wedge^{g+1}\mathbf{C}^{2g+2} \to \wedge^{g+1}\mathbf{C}^{2g+2}$ whose eigenspaces are so-invariant. These are the subspaces we have to determine.

Note $e_S \wedge e_T \neq 0$ iff $T = S'$, the complement of S. One easily verifies that if $1 \in S$ we have:

$$e_S \wedge e_{S'} = sgn(\sigma_S) e_1 \wedge e_2 \wedge \ldots e_{2g+2}, \qquad \text{so} \quad C(e_S) = sgn(\sigma_S)\epsilon_{S'}.$$

As $e_{S'} \wedge e_S = (-1)^{g+1} e_S \wedge e_{S'}$, we get $C(e_{S'}) = sgn(\sigma_S)(-1)^{g+1}\epsilon_S$ (with $1 \in S$). Therefore

$$A(e_S) = sgn(\sigma_S) e_{S'}, \qquad A(e_{S'}) = sgn(\sigma_S)(-1)^{g+1} e_S \qquad \text{if} \quad 1 \in S.$$

The eigenvalues of A are thus $\pm i^{g+1}$ and the eigenvectors are $e_S \pm sgn(\sigma_S) i^{g+1} e'_S$. QED

3.3.6 A quartic polynomial

We consider the g-dimensional subspaces of $\mathbf{P}^{2g+1}$ which are tangent to Q_z:

$$\bar{B}_z := \left\{ \mathbf{P}^g \subset \mathbf{P}^{2g+1} : \ rank(Q_z \text{ restricted to } \mathbf{P}^g) \leq g \right\} \qquad (\subset Gr_{SL}).$$

The subvariety $\bar{B}_z$ is defined by a quadratic polynomial P_z in the Plücker coordinates, that is by a section in $H^0(Gr_{SL}, \mathcal{O}_{Gr_{SL}}(2))$ ([vG], Thm 3):

$$\bar{B}_z = Z(P_z), \qquad P_z \in H^0(Gr_{SL}, \mathcal{O}_{Gr_{SL}}(2)) \subset S^2(\wedge^{g+1}\mathbf{C}^{2g+2}).$$

This section is in fact the unique SO_z-invariant in $H^0(Gr_{SL}, \mathcal{O}_{Gr_{SL}}(2))$ and will be determined explicitly in the lemma below. In case $g = 2$ the quartic surface in $\mathbf{P}V$ defined by P_z is the Kummer surface of the curve $\mathcal{C}_z$ ([DR]).

3.3.7 Lemma. The variety $\bar{B}_z$ is defined by

$$P_z = \sum_S z_S e_S^2,$$

where S runs over the subsets with $g+1$ elements of $B = \{1, \ldots, 2g+2\}$ and

$$z_S := z_{i_1} z_{i_2} \ldots z_{i_{g+1}}, \qquad e_S = e_{i_1} \wedge \ldots \wedge e_{i_{g+1}} \qquad \text{if} \quad S = \{i_1, \ldots, i_{g+1}\}.$$

Proof. From [vG], Theorem 3′ and its proof we know that the trivial representation of so_z has multiplicity one in $H^0(Gr_{SL}, \mathcal{O}_{Gr_{SL}}(2)) = V(2\lambda_{g+1})$. This sl_{2g+2} representation corresponds to the partition μ: $2g+2 = 2+2+\ldots+2$ and is thus realized as $(\mathbf{C}^{2g+2})^{\otimes 2g+2} c_\mu$ where c_μ is the Young symmetrizer, one has in fact $V(2\lambda_{g+1}) \subset S^2(\wedge^{g+1}\mathbf{C}^{2g+2})$ (see [FH], §15.5, p. 233-236). We use the standard tableau as in [vG] (and [FH]) to construct c_μ.

The trivial so_z sub-representation is obtained as follows. The quadratic form Q_z defines the so_z-invariant tensor $\sum_{i=1}^{2g+2} z_i e_i \otimes e_i \in \mathbf{C}^{2g+2} \otimes \mathbf{C}^{2g+2}$, and we consider the $g+1$-fold tensor product of this tensor, where we insert the k-th factor in the positions k and $g+1+k$:

$$\tau_z := \sum_{i_1, \ldots, i_{g+1}=1}^{2g+2} z_{i_1} z_{i_2} \ldots z_{i_{g+1}} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{g+1}} \otimes e_{i_1} \otimes \ldots e_{i_{g+1}} \quad (\in (\mathbf{C}^{2g+2})^{\otimes 2g+2}).$$

Thus τ_z is so_z invariant and it has the advantage that $\tau_z c_\mu$ is easily determined:

$$\tau_z c_\mu = (cst) \sum_S z_S \, e_S \cdot e_S,$$

with (cst) a non-zero integer and where we sum over all subsets S with $g+1$ elements of $\{1, \ldots, 2g+2\}$ (In fact, $c_\mu = a_\mu b_\mu$ and a_μ symmetrizes the indices $1,\ g+2;\ 2,\ g+3;\ \ldots;\ g+1,\ 2g+2$ of a tensor (but τ_z is already symmetric in these indices) and next b_μ antisymmetrizes the first $g+1$ and last $g+1$ indices, giving the result above). QED

3.4 Verifying the contraction criterium

3.4.1

We need to determine the images of the Ω_{ij}'s in the representation $S^2(\wedge^{g+1}\mathbf{C}^{2g+2})$. In the standard representation $\mathbf{C}^{2g+2}$ of $so(2g+2)$, with basis e_i and dual basis ϵ_i, we have:

$$F_{ij} = 2(e_i \otimes \epsilon_j - e_j \otimes \epsilon_i) \qquad \in (\mathbf{C}^{2g+2}) \otimes (\mathbf{C}^{2g+2})^* = End(\mathbf{C}^{2g+2}).$$

On $\wedge^{g+1}\mathbf{C}^{2g+2}$, with basis $e_S = e_{i_1} \wedge \ldots \wedge e_{i_{g+1}}$ as before, F_{ij} acts as

$$F_{ij}(e_{i_1} \wedge \ldots \wedge e_{i_{g+1}}) = F_{ij}(e_{i_1}) \wedge e_{i_2} \wedge \ldots \wedge e_{i_{g+1}} + \ldots + e_{i_1} \wedge \ldots \wedge e_{i_g} \wedge F_{ij}(e_{i_{g+1}}).$$

Define a subset of $B = \{1, \ldots, 2g+2\}$ by

$$S + ij = (S \cup \{i,j\}) - (S \cap \{i,j\}).$$

Then $F_{ij}(e_S)$ is, up to sign, $2e_{S+ij}$ if $|S\cap\{i,j\}|=1$ and is zero otherwise:

$$F_{ij}=2\sum_{S,\,|S\cap\{i,j\}|=1} t_{S,ij}e_{S+ij}\otimes\epsilon_S$$

with $t_{S,ij}=\pm 1$. Note that if $|S\cap\{i,j\}|=1$ then $F_{ij}^2 e_S=-4e_S$ so $t_{S,ij}=-t_{S+ij,ij}$.

3.4.2 Lemma. Let W be a complex vector space, let $D\in W\otimes W^*$ be a linear differential operator and let $P\in S^kW$. Then the contraction of P with the symbol of D^2 is given by:

$$\sigma(D^2)\cdot P=2D(P)D\qquad\in S^{k-1}W\otimes W^*.$$

Proof. We view elements of W as linear forms in variables X_a. Then elements of W^* are linear operators with constant coefficients and $D=\sum_a f_a\partial_a$ with $f_a\in W$. The symbol of D^2 is then:

$$\sigma(D^2)=\sum_{a,b} f_af_b\partial_a\partial_b\qquad\in S^2W\otimes S^2W^*.$$

Recall that we use the convention (see 2.2.4):

$$\partial_a\partial_b:=\partial_a\otimes\partial_b+\partial_b\otimes\partial_a$$

for elements in S^2W^*. The contraction of a polynomial $P\in S^kW$ with $\partial_a\partial_b\in S^2W^*$ is thus:

$$(\partial_a\partial_b)\cdot P=(\partial_bP)\partial_a+(\partial_aP)\partial_b\qquad\in S^{k-1}W\otimes W^*.$$

Therefore

$$\begin{aligned}\sigma(D^2)\cdot P&=\textstyle\sum_{a,b}f_af_b\left((\partial_bP)\partial_a+(\partial_aP)\partial_b\right)\\&=\textstyle 2\sum_a f_a(\sum_b f_b(\partial_bP))\partial_a\\&=2D(P)D.\end{aligned}$$

QED

3.4.3 Theorem. The composition, still denoted by P_z:

$$\mathcal{P}\longrightarrow S^2\wedge^{g+1}\mathbf{C}^{2g+2}\longrightarrow S^4V(\omega_{g+1}),\qquad z\longmapsto P_z,$$

with $P_z=\sum_S z_Se_S^2$ (as in Subsection 3.3.6), satisfies the differential equations:

$$\left(\frac{1}{16}\sum_{j\neq i}\frac{\sigma\tilde\rho(\Omega_{ij})}{z_i-z_j}\right)\cdot P_z+(\partial_{z_i}P_z)E=0\qquad\qquad(1\le i\le 2g+2)$$

with $E=\sum e_S\otimes\epsilon_S$ the Euler vector field and $\tilde\rho:U(so(2g+2))\to\mathrm{End}(S^2(\wedge^{g+1}\mathbf{C}^{2g+2}))$. Thus P_z is a horizontal section for the connection ∇_λ^W with $W=S^4V(\omega_{g+1})$ and $\lambda=-16$.

Proof. For symmetry reasons it suffices to verify the equation for $i=1$. By Lemma 3.4.2

$$\sigma\tilde{\rho}(\Omega_{1j})\cdot e_S^2 = 2\rho(F_{1j})(e_S^2)\rho(F_{1j}) = 4e_S\rho(F_{1j})(e_S)\rho(F_{1j}).$$

To simplify notation we write Ω_{ij} for $\sigma\tilde{\rho}(\Omega_{ij})$ and F_{ij} for $\rho(F_{ij})$ (as in 3.4.1). Then $F_{1j}(e_S)=0$ unless $|S\cap\{1,j\}|=1$. Assume $1\in S$ and $j\notin S$ (so $j\in B-S$) and consider the contraction of Ω_{1j} with the term $z_Se_S^2+z_{S+ij}e_{S+ij}^2$ from P_z:

$$\begin{aligned}\Omega_{1j}\cdot(z_Se_S^2+z_{S+1j}e_{S+1j}^2) &= 8(t_{S,1j}z_S+t_{S+1j,1j}z_{S+1j})e_Se_{S+1j}F_{1j}\\ &= 8t_{S,1j}(z_1z_{\bar{S}}-z_jz_{\bar{S}})e_Se_{S+1j}F_{1j}\\ &= (z_1-z_j)\Omega_{1j}\cdot z_{\bar{S}}e_S^2,\end{aligned}$$

where $\bar{S}:=S-\{1\}$. Therefore we get:

$$\frac{\Omega_{1j}}{z_1-z_j}\cdot P_z = \Omega_{1j}\cdot\left(\sum_{S\ni 1,\, j\notin S} z_{\bar{S}}e_S^2\right).$$

Summing this result over all j and changing the order of summation gives:

$$\begin{aligned}\left(\textstyle\sum_{j\neq 1}\frac{\Omega_{1j}}{z_1-z_j}\right)\cdot P_z &= \textstyle\sum_{S\ni 1} z_{\bar{S}}\left(\sum_{j\in B-S}\Omega_{1j}\cdot e_S^2\right)\\ &= 8\textstyle\sum_{S\ni 1} z_{\bar{S}}e_S\left(\sum_{j\in B-S}t_{S,1j}e_{S+1j}F_{1j}\right)\end{aligned}$$

where, as before, $B=\{1,\ldots,2g+2\}$. On the other hand, since $P_z=\sum z_Se_S^2$, we have:

$$\partial_{z_1}P_z = \sum_{S\ni 1} z_{\bar{S}}e_S^2.$$

Thus the theorem follows if we prove, for all S with $1\in S$:

$$2e_SE + \sum_{j\in B-S} t_{S,1j}e_{S+1j}F_{1j} = 0.$$

With the definition of F_{1j} we find:

$$\begin{aligned}\textstyle\sum_{j\in B-S}t_{S,1j}e_{S+1j}F_{1j} &= \textstyle\sum_{j\in B-S}t_{S,1j}e_{S+1j}\left(\sum_{T,\,|T\cap\{1,j\}|=1}2t_{T,1j}e_{T+1j}\otimes\epsilon_T\right)\\ &= 2\textstyle\sum_T\left(\sum_{j\in B-S,\,|T\cap\{1,j\}|=1}t_{S,1j}t_{T,1j}e_{S+1j}e_{T+1j}\right)\otimes\epsilon_T.\end{aligned}$$

Note that $e_SE=\sum_T e_Se_T\otimes\epsilon_T$ (with $T\in B$, $|T|=g+1$). Comparing coefficients of ϵ_T, it remains to prove that (for all $S\ni 1$ and all T):

$$e_Se_T + \sum_{j\in B-S,\,|T\cap\{1,j\}|=1} t_{S,1j}t_{T,1j}e_{S+1j}e_{T+1j} = 0.$$

We show that the relations for $1\in T$ follow from those with $1\notin T$. In fact, we only want this relation in $V(4\omega_{g+1})$, so we restrict ourselves to the subspace $V(2\omega_{g+1})\subset\wedge^{g+1}\mathbf{C}^{2g+2}$. The kernel of the restriction map is $\langle \ldots, e_T - sgn(e_T) i^{g+1} e_{T'}, \ldots \rangle$ (or the other space in Lemma 3.3.5; the argument we give leads to the same conlusion in both cases).

Consider a T with $1 \in T$ and $j \notin T$. Then $1 \notin T + 1j$ and after restriction:

$$e_T = sgn(\sigma_T) i^{g+1} e_{T'}, \qquad e_{T+1j} = sgn(\sigma_{(T+1j)'}) i^{-(g+1)} e_{(T+1j)'}$$

where $'$ stands for the complement in B. Since $(T+1j)' = T' + 1j$, substituting these relations and multiplying throughout by i^{g+1} we get:

$$(-1)^{g+1} sgn(\sigma_T) e_S e_{T'} + \sum_{j \in B-S,\ |T \cap \{1,j\}|=1} t_{S,1j} t_{T,1j} sgn(\sigma_{T'+1j}) e_{S+1j} e_{T'+1j}$$

Next we observe that

$$sgn(\sigma_{T'+1j}) = (-1)^{g+1} t_{T,1j} t_{T',1j} sgn(\sigma_T),$$

so it suffices to consider the relations with $1 \notin T$. Let $T := \{1 = i_1, \ldots, i_{g+1}\}$, $T' := \{j_1, \ldots, j_{g+1}\}$ with $j = j_k$ and $i_l < j < j_{l+1}$. To get $\sigma_{T'+1j}$, first apply the permutation $(1\ g+2) \ldots (g+1\ 2g+2)$ to B, the sign of this permutation is $(-1)^{g+1}$. Then apply σ_T. Next apply the cyclic permutation $(j_1\ \ldots j_{g+1}\ 1)$ (with sign $(-1)^{g+1}$) and finally apply the inverse of $(j_k\ \ldots j_{g+1}\ i_2\ \ldots i_l)$ (with sign $(-1)^{g-k+l}$). The resulting permutation is $\sigma_{T'+1j}$ and has sign $(-1)^{g+k+l} sgn(\sigma_T)$. On the other hand, since $1 \in T$, $j \notin T$ we get $F_{1j}(e_T) = -2(-1)^{l-1} e_{T+1j}$, so $t_{T,1j} = (-1)^l$ and $F_{1j}(e_{T'}) = +2(-1)^{k-1} e_{T'+1j}$, so $t_{T',1j} = (-1)^{k-1}$. This gives the formula for $sgn(\sigma_{T'+1j})$.

¿From now on we consider only T's with $1 \notin T$ and S with $1 \in S$. We show that the desired relation:

$$e_S e_T + \sum_{i \in (B-S) \cap T} t_{S,1i} t_{T,1i} e_{S+1i} e_{T+1i}.$$

is a Plücker relation. Thus it holds in $H^0(Gr_{SL}, \mathcal{O}(2))$ and therefore also upon restriction to $Gr_{SO} \subset Gr_{SL}$.

Let $S = \{1 = i_1, \ldots, i_{g+1}\}$ and consider the Zariski open subset $U \subset Gr_{SL}$ of $g+1$-dimensional subspaces $W \subset \mathbf{C}^{2g+2}$ with Plücker coordinate $p_S \neq 0$. Any such W has a (unique) basis $\{\ldots, w_i, \ldots\}$ with:

$$W = \langle w_1, \ldots, w_{g+1} \rangle, \qquad \text{and} \quad (w_k)_{i_l} = \delta_{kl}, \qquad (S = \{1 = i_1, \ldots, i_{g+1}\},\ i_1 < \ldots < i_{g+1})$$

where δ_{kl} is Kronecker's delta. Let M be the $(g+1) \times (2g+2)$ matrix whose rows are the w_i. Let M_i be the i-th column of M_W (note $M_{i_j} = f_j$, the j-th standard basis vector of $\mathbf{C}^{g+1}$). Then the Plücker coordinate $p_{k_1,\ldots,k_{g+1}}$ is the determinant of the $(g+1) \times (g+1)$ submatrix of M whose j-th column is M_{k_j}, we write:

$$p_{k_1,\ldots,k_{g+1}} = det(M_{k_1}, \ldots, M_{k_{g+1}}) \qquad \text{with} \quad k_1 < \ldots < k_{g+1}.$$

Let $j \notin S$, with $i_l < j < i_{l+1}$. Then $p_{S+1j} = (-1)^{l-1} (M_j)_1$ since

$$det(f_{i_2}, \ldots, f_{i_l}, M_j, f_{i_{l+1}}, \ldots, f_{g+1}) = (-1)^{l-1} det(M_j, f_{i_2}, \ldots, f_{i_l}, f_{i_{l+1}}, \ldots, f_{g+1}) = (-1)^{l-1} (M_j)_1.$$

As earlier, since $1 \in S$ we have $t_{S,1j} = -(-1)^{l-1}$, hence $p_{S+1j} = -t_{S,1j}(M_j)_1$.

Let $T = \{j_1, \ldots, j_{g+1}\}$, let $T \cap S' = \{a_1, \ldots, a_q\}$ and let $j = j_n = a_k$. Then $p_T = det(M_{j_1}, \ldots, M_{j_{g+1}})$ is, up to sign, the determinant of a $q \times q$ submatrix N of the $(g+1) \times q$ matrix with columns $M_{a_1}, \ldots, M_{a_q}$, we write $p_T = t \cdot det(N)$ with $t = \pm 1$. Then

$$\begin{aligned} p_{T+1j} &= det(f_1, M_{j_1}, \ldots, \widehat{M_{j_n}}, \ldots, M_{j_{g+1}}) \\ &= (-1)^{j_n - 1} det(M_{j_1}, \ldots, M_{j_{n-1}}, f_1, M_{j_{n+1}}, \ldots, M_{j_{g+1}}) \\ &= (-1)^{j_n-1}(-1)^{k+1} t \cdot det(N^{1k}), \end{aligned}$$

where N^{1k} is the $(q-1) \times (q-1)$ submatrix of N obtained by deleting the first row and k-th column. Since $1 \notin T$, we have $t_{T,1j} = (-1)^{j_n-1}$ and thus $p_{T+1j} = t(-1)^{k+1} det(N^{1k})$.

Substituting these expressions for p_{S+1j} and p_{T+1j} we get:

$$1 \cdot t \cdot det(N) + \sum_{j \in \{a_1, \ldots, a_q\}} t(-1)^k (M_j)_1 det(N^{1k})$$

which is zero in virue of a well known formula for the determinant. QED

4 The Heisenberg group and the Spin representation

4.1

We recall the basic facts on the Heisenberg group and we discuss the projective representation of its automorphism group. In Subsection 4.4 we relate the Heisenberg group (in its irreducible 2^g-dimensional representation) with the (half) spin representation of the orthogonal group. Combining this with previous results, we can finally write down Hitchin's connection in Subsection 4.5.

4.2 The Heisenberg group

4.2.1 Definitions

We introduce a variant of the Heisenberg group (cf. Subsection 2.5.2). For any positive integer g we define a (finite) Heisenberg group G by:

$$G = G_g := \{(t, x) : \ t \in \mathbf{C}, \ t^4 = 1, \quad x = (\xi, \xi') \in \mathbf{F}_2^g \times \mathbf{F}_2^g\}$$

with identity element $(1, 0)$ and multiplication law:

$$(t, (\xi, \xi'))(s, (\eta, \eta')) := (ts(-1)^{\xi\eta'}, \xi + \eta, \xi' + \eta')$$

with $\xi\eta' := \sum_{i=1}^g \xi_i \eta_i'$. The group G has 2^{2g+2} elements, it is non-abelian, in fact its center is $\{(t, 0)\}$. The inverse of (t, x) can be found as follows:

$$(t, (\xi, \xi'))(s, (\xi, \xi')) = ((-1)^{\xi\xi'} ts, (0, 0)), \qquad \text{thus} \quad (t, x)^{-1} = ((-1)^{\xi\xi'} t^{-1}, x),$$

So for any non-zero x there is are elements $(t,x) \in G$ of order 2 (and also of order 4).

We define a symplectic form on $\mathbf{F}_2^{2g}$ by:

$$E(x,y) = \xi\eta' + \eta\xi', \qquad \text{with} \quad x = (\xi,\xi'),\ y = (\eta,\eta')$$

thus $E(x,x) = 0$ for all x but E is non-degenerate. Note that E is related to the commutator in G:

$$(t,x)(s,y)(t,x)^{-1}(s,y)^{-1} = ((-1)^{E(x,y)},0).$$

4.2.2 Representations.

Let V be the 2^g dimensional vector space of complex valued functions $\mathbf{F}_2^g \to \mathbf{C}$. It has a standard basis consisting of δ-functions

$$X_\sigma : \mathbf{F}_2^g \longrightarrow \mathbf{C}, \qquad X_\sigma(\sigma) = 1, \quad X_\sigma(\rho) = 0 \quad \text{if } \sigma \neq \rho.$$

The Heisenberg group G has a representation U on V, the Schrödinger representation:

$$(U(t,(\xi,\xi'))f)(\sigma) := t(-1)^{\sigma\xi'}f(\sigma+\xi), \qquad \text{thus} \quad U(t,(\xi,\xi'))X_\sigma = t(-1)^{(\sigma+\xi)\xi'}X_{\sigma+\xi}.$$

For every $x \in \mathbf{F}_2^{2g} - \{0\}$ choose a $(t_x,x) \in G$ of order two. We define:

$$U_x := U(t_x,x) \qquad (\in GL(V)), \qquad \text{so} \quad U_x^2 = I$$

and define $U_0 = I$. Then $Im(U) = \{tU_x :\ t^4 = 1,\ x \in \mathbf{F}_2^{2g}\}$.

4.2.3 Automorphisms.

We define a subgroup of $Aut(G)$ by:

$$A(G) := \{\phi \in Aut(G) :\ \phi((t,0)) = (t,0) \quad \forall t\}.$$

The elements of $A(G)$ are the automorphisms which are the identity when restricted to the center of G. For $\phi \in A(G)$ and $(t,x) \in G$ we can then write:

$$\phi(t,x) := (f_\phi(x)t, M_\phi(x)), \qquad \text{with} \quad M_\phi : \mathbf{F}_2^{2g} \longrightarrow \mathbf{F}_2^{2g}, \quad f_\phi : \mathbf{F}_2^{2g} \longrightarrow \mathbf{C}^*$$

(note $\phi(t,x) = \phi(t,0)\phi(1,x) = (t,0)\phi(1,x)$). The map $M : A(G) \to Aut(\mathbf{F}_2^{2g})$, $\phi \mapsto M_\phi$ is a homomorphism.

Assume that $\phi \in \ker(M)$, thus M_ϕ is the identity. Then one verifies that f_ϕ is a homomorphism so we must have $f_\phi(x) = (-1)^{E(x,y)}$ for some $y = y_\phi \in \mathbf{F}_2^{2g}$. But then ϕ is an interior automorphism since also

$$Int_y : (t,x) \longmapsto (1,y)(t,x)(1,y)^{-1} = ((-1)^{E(x,y)}t,x).$$

Thus $\ker(M) \cong G/Center(G) \cong \mathbf{F}_2^{2g}$.

Since automorphisms preserve commutators in G, the image of M lies in $Sp(2g,\mathbf{F}_2) = Sp(\mathbf{F}_2^{2g},E)$. There is an exact sequence:

$$0 \longrightarrow \mathbf{F}_2^{2g} \longrightarrow A(G) \xrightarrow{M} Sp(2g,\mathbf{F}_2) \longrightarrow 0.$$

(See Theorem 4.2.6 below for the surjectivity of M.)

4.2.4 A projective representation.

The Schrödinger representation U of G on V is the unique irreducible representation of G in which $(t,0)$ acts by mutiplication by t. Given $\phi \in A(G)$, the representation $U \circ \phi$ enjoys the same property. Hence by Schur's lemma we get a linear map, unique up to scalar multiple,

$$\tilde{T}_\phi : V \longrightarrow V, \qquad \text{with} \quad \tilde{T}_\phi U(h) = U(\phi(h))\tilde{T}_\phi,$$

for all $h \in G$. In this way we get a projective representation of $A(G)$:

$$\tilde{T} : A(G) \longrightarrow PGL(V), \qquad \phi \longmapsto \tilde{T}_\phi.$$

Note that we may take $\tilde{T}_\phi = U_y$ when $\phi = Int_y$.

On the other hand, as $U : G \to GL(V)$ is injective we have $Im(U) \cong G$. Any $T \in GL(V)$ which normalizes $Im(U)$ thus defines an automorphism ϕ_T of G (if $TU(h)T^{-1} = U(h')$ then $\phi_T(h) := h'$). Since the center of G acts by scalar multiples of the identity, we have $\phi_T \in A(G)$. Thus we get an exact sequence

$$0 \longrightarrow \mathbf{C}^* \longrightarrow Normalizer_{GL(V)}(Im(U)) \longrightarrow A(G) \longrightarrow 0.$$

4.2.5 Definition

For $x = (\xi, \xi') \in \mathbf{F}_2^{2g} - \{0\}$, we define the transvection

$$T_x : \mathbf{F}_2^{2g} \longrightarrow \mathbf{F}_2^{2g}, \qquad y \longmapsto y + E(y,x)x.$$

Then $T_x \in Sp(2g, \mathbf{F}_2)$ and the (finite) symplectic group $Sp(2g, \mathbf{F}_2)$ is generated by transvections. Note that the transvections are involutions: $T_x^2 = 1$.

4.2.6 Theorem. For $x \in \mathbf{F}_{2g}^2 - \{0\}$ let

$$\tilde{T}_x := U_x + iI, \qquad \text{with} \quad i^2 = -1.$$

Then $\tilde{T}_x \in A(G)$ and

$$M(\tilde{T}_x) = T_x \qquad \text{that is:} \qquad \tilde{T}_x U_y \tilde{T}_x^{-1} = t_{x,y} U_{T_x(y)},$$

for all $y \in \mathbf{F}_2^{2g}$ and some $t_{x,y} \in \mathbf{C}$ with $t_{x,y}^4 = 1$. Therefore the homomorphism $M : A(G) \to Sp(2g, \mathbf{F}_2)$ is surjective.

Moreover, in $A(G)$ we have $\tilde{T}_x^2 = Int_x$.

Proof. Since $U_x^2 = I$, the eigenvalues of U_x are ± 1. Thus $U_x + iI$ is invertible (its inverse is $(1/2)(U_x - iI)$). In case $E(x,y) = 0$, U_x and U_y commute and $T_x(y) = y$, thus the relation holds. In case $E(x,y) = 1$, $U_x U_y = -U_y U_x$ and $T_x(y) = T_{x+y}$, and the relation holds because:

$$\begin{aligned}
\tilde{T}_x U_y \tilde{T}_x^{-1} &= (1/2)\left(U_x U_y U_x - i(U_x U_y - U_y U_x) + U_y\right) \\
&= (1/2)(-U_y - 2iU_x U_y + U_y) \\
&= \pm U_{x+y} \qquad (\in Im(U)),
\end{aligned}$$

in fact, U_{x+y} and $-iU_xU_y \in Im(U)$ differ by a scalar multiple t and since both elements have order two, $t = \pm 1$. Thus $\tilde{T}_x$ normalizes $Im(U)$ and defines an element of $A(G)$ indicated by the same symbol. The relation shows that $M(\tilde{T}_x) = T_x$. Since the T_x generate $Sp(2g, \mathbf{F}_2)$ the map M is surjective.

Finally $(U_x + iI)^2 = U_x^2 + 2iU_x - I = 2iU_x$, and conjugation by U_x induces Int_x, which proves the last statement. QED

4.2.7 Example

A particular case is when $x = (0, \xi')$ with $\xi' = (1, 0, \ldots, 0) \in \mathbf{F}_2^g$. Then on the basis $X_{(0,\ldots,0)}, \ldots, X_{(0,\tau)}, \ldots, X_{(1,0,\ldots,0)}, \ldots, X_{(1,\tau)}, \ldots$ with $\tau \in \mathbf{F}_2^{g-1}$ we have:

$$U_x = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \tilde{T}'_x := \tfrac{1}{1+i}\tilde{T}_x = \begin{pmatrix} I & 0 \\ 0 & iI \end{pmatrix},$$

note that $\tilde{T}'_x$ and $\tilde{T}_x$ define the same element of $A(G)$.

4.3 Notation

4.3.1

In dealing with hyperelliptic curves and the half-spin representation of the orthogonal group, the following (classical) notation for points in $\mathbf{F}_2^{2g}$ is convenient ([DO], VIII.3; [M]). Let

$$B := \{1,\, 2, \ldots, 2g+2\}, \qquad \text{then} \quad F_B := \left\{ f : B \to \mathbf{F}_2 :\ \sum_{b \in B} f(b) = 0 \right\} \Big/ \{0, [b \mapsto 1]_{b \in B}\}.$$

is an $\mathbf{F}_2$-vector space of dimension $2g$. For a subset $T \subset B$ with an even number elements we denote by $x_T \in F_B$ (or simply T) the element defined by the function f with $f(b) = 1$ iff $b \in T$. Note that $x_T = x_{T'}$ when T' is the complement of T in B, moreover

$$x_T + x_S = x_R \qquad \text{with} \quad R = T + S := (T \cup S) - (T \cap S).$$

4.3.2

We fix the following isomorphism of $\mathbf{F}_2$-vector spaces, and identify them in this way in the remainder of the paper ([DO], VIII.3, Lemma 2, but note we interchanged $(e_i, 0) \leftrightarrow (0, e_i)$):

$$F_B \xrightarrow{\cong} \mathbf{F}_2^g \times \mathbf{F}_2^g, \qquad x_{\{2i-1,\,2i\}} = (0, e_i), \quad x_{\{2i,\,2i+1,\ldots,2g+1\}} = (e_i, 0) \qquad (1 \le i \le g).$$

The symplectic form E on $\mathbf{F}_2^g \times \mathbf{F}_2^g$ can be now be easily computed on F_B by:

$$E(x_T, x_S) = |T \cap S| \mod 2, \qquad \text{thus} \quad E(x_{ij}, x_{kl}) = \begin{cases} 0 & \text{if} \quad i,\, j,\, k,\, l \quad \text{are distinct,} \\ 1 & \text{if} \quad i < j = k < l \end{cases}$$

where we write x_{ij} for $x_{\{i,j\}}$ etc. For $g=1$ we have

$$x_{12}=x_{34}=(1,0),\quad x_{13}=x_{24}=(1,1),\quad x_{14}=x_{23}=(0,1).$$

For $g=2$ one has 15 non-zero points $x_{ij}=x_{klmn}$ when $\{i,j,k,l,m,n\}=B$:

$$x_{12}=x_{3456}=((0,0),(1,0)),\quad x_{2345}=x_{16}=((1,0),(0,0)),\quad x_{26}=x_{1345}=((1,0),(1,0)).$$

4.4 The Spin representation

4.4.1

The two half spin representations of $so(2g+2)$ are each realized on a 2^g-dimensional vector space. We recall, using the Clifford algebra, how they can be constructed using the Heisenberg group. We will have to consider the Heisenberg group G_{g+1} which acts on a 2^{g+1}-dimensional vector space. Elements of order two in the Heisenberg group will define the spin representation of $so(2g+2)$. Restriction to suitable subspaces will give the half spin representations and their relation with G_g (a subquotient of G_{g+1}). This relation between $so(2g+2)$ and the Heisenberg group is used in 4.5.3 to prove the Heisenberg invariance of the flat connections introduced in 3.2.4.

4.4.2

To accomodate both G_g and G_{g+1} we extend the construction of §4.3.1. The inclusion

$$B:=\{1,\dots,2g+2\}\hookrightarrow B^\sharp:=\{1,\dots,2g+4\}$$

and extension by zero of functions on B to functions on $B^\sharp$ induces

$$\tilde{F}_B:=\left\{f:B\to\mathbf{F}_2:\sum_{b\in B}f(b)=0\right\}\hookrightarrow\tilde{F}_{B^\sharp}:=\left\{f:B\to\mathbf{F}_2:\sum_{b'\in B^\sharp}f(b')=0\right\}.$$

The function $g'\in\tilde{F}_{B^\sharp}$ with $g'(b')=1$ (all $b'\in B^\sharp$) does not lie in $\tilde{F}_B$, thus

$$\tilde{F}_B\hookrightarrow F_{B^\sharp}:=\tilde{F}_{B^\sharp}/\{0,g'\}.$$

The image in $F_{B^\sharp}$ of the function g ($\in\tilde{F}_B$) with $g(b)=1$ (all $b\in B$) is the element

$$p':=x_{\{1,2,\dots,2g+2\}}=x_{\{2g+3,2g+4\}}\in F_{B^\sharp}.$$

Using the definition of the symplectic form on $F_{B^\sharp}$ (§4.3.2), which we denote by $E^\sharp$, one finds that:

$$\tilde{F}_B=(p')^\perp,\qquad\text{with}\quad p'^\perp:=\{x\in F_{B^\sharp}:\ E^\sharp(x,p')=0\,\}\quad\text{and}\quad F_B=(p')^\perp/\{0,p'\}.$$

From §4.3.2 we have an identification:

$$F_{B^\sharp}=\mathbf{F}^{g+1}\times\mathbf{F}^{g+1},\qquad p'=(0,(1,1,\dots,1))\qquad(\in F_{B^\sharp}).$$

The Heisenberg group defined by this identification (cf. §4.2.1) will be denoted by $G^\sharp$, its Schrödinger representation by $U^\sharp$ (on the 2^{g+1}-dimensional vector space $V^\sharp := \{f : \mathbf{F}_2^{g+1} \to \mathbf{C}\}$ with basis of δ-functions $Y_{\sigma'}$, $\sigma' \in \mathbf{F}_2^{g+1}$).

For any $x' \in (p')^\perp$ ($\subset F_{B^\sharp}$), the maps $U^\sharp_{p'}$ and $U^\sharp_{x'}$ commute. Therefore the $U^\sharp_{x'}$ with $x' \in (p')^\perp$ act on the two eigenspaces of $U^\sharp_{p'}$ which are:

$$V_+^\sharp := \langle \dots, Y_{\sigma'}, \dots \rangle_{\sigma'_1+\dots+\sigma'_{g+1}=0}, \qquad V_-^\sharp := \langle \dots, Y_{\sigma'}, \dots \rangle_{\sigma'_1+\dots+\sigma'_{g+1}=1}.$$

A quotient map from $(p')^\perp$ ($\subset (\mathbf{F}_2^g \times \mathbf{F}_2) \times (\mathbf{F}_2^g \times \mathbf{F}_2)$) to $F_B = \mathbf{F}_2^g \times \mathbf{F}_2^g$ with kernel $\{0, p'\}$ is:

$$(p')^\perp \longrightarrow F_B = (p')^\perp/\{0,p'\} \cong \mathbf{F}_2^{2g}, \qquad x' := ((a, a_{g+1}), (b, b_{g+1})) \longmapsto x := (a, \bar b)$$

with $\bar b = (b_1 + b_{g+1}, \dots, b_g + b_{g+1})$. We define an isomorphism of vector spaces:

$$V_+^\sharp \longrightarrow V, \qquad Y_{\sigma_1,\dots,\sigma_g,\sigma_{g+1}} \longmapsto X_{\sigma_1,\dots,\sigma_g}.$$

4.4.3 Lemma. For any $x' \in (p')^\perp$ mapping to $x \in F_B$ there is a commutative diagram:

$$\begin{array}{ccc} V_+^\sharp & \xrightarrow{\cong} & V \\ {\scriptstyle U^\sharp(t,x')}\downarrow & & \downarrow{\scriptstyle U(t,x)} \\ V_+^\sharp & \xrightarrow{\cong} & V \end{array}$$

Moreover, if $1 \le j, k \le 2g+1$, then restriction of $U^\sharp_{x_{jk}}$ to $V_+^\sharp \cong V$ is given by $\pm U_{x_{ij}}$.

Proof. The two compositions along the square are (with a, b, $\sigma \in \mathbf{F}_2^g$):

$$Y_{\sigma,\sigma_{g+1}} \longmapsto X_\sigma \longmapsto (t,(a,\bar b))X_\sigma = t(-1)^{(a+\sigma)\bar b} X_{a+\sigma}, \qquad \text{and}$$

$$\begin{aligned} Y_{\sigma,\sigma_{g+1}} &\longmapsto (t,(a,a_{g+1}),(b,b_{g+1}))Y_{\sigma,\sigma_{g+1}} \\ &= t(-1)^{(a+\sigma)b+(a_{g+1}+\sigma_{g+1})b_{g+1}} Y_{a+\sigma_g, a_{g+1}+\sigma_{g+1}} \\ &\longmapsto t(-1)^{(a+\sigma)b+(a_{g+1}+\sigma_{g+1})b_{g+1}} X_{a+\sigma}. \end{aligned}$$

Recall $E^\sharp(x',p') = 0$, so $a_{g+1} = \sum_{i=1}^g a_i$. As $Y_{\sigma,\sigma_{g+1}} \in V_+^\sharp$ we also have $\sigma_{g+1} = \sum_{i=1}^g \sigma_i$, thus:

$$(a+\sigma)b + (a_{g+1}+\sigma_{g+1})b_{g+1} = \left(\sum_{i=1}^g (a_i+\sigma_i)b_i\right) + \left(\sum_{i=1}^g (a_i+\sigma_i)b_{g+1}\right) = (a+\sigma)\bar b,$$

which shows that the diagram commutes. The last statement follows from the fact that such a $x' = x_{jk}$ lies in $(p')^\perp$ ($\subset F_{B^\sharp}$) and that the homomorphism $x' \mapsto x$ ($\in F_B$) maps x_{jk} to x_{jk} (use 4.3.2). Since $U^\sharp_{x_{jk}} = U^\sharp(t', x')$, $U_{x_{jk}} = U(t,x)$ for some t', $t \in \mathbf{C}$ choosen such that each transformation has order two, the statement follows from the commutativity of the diagram. QED

4.4.4 The Clifford algebra

The Clifford algebra $C(Q)$ of the quadratic form $Q = x_1^2 + \ldots + x_{2g+2}^2$ on $V_Q = \mathbf{C}^{2g+2}$ is the quotient of the tensor algebra $T(V_Q)$ by the two-sided ideal I generated by the elements $v \otimes v - Q(v)$ for $v \in V_Q$ ([FH], §20.1):

$$C(Q) := T(V_Q)\big/I = (\mathbf{C} \,\oplus\, V_Q \,\oplus\, V_Q \otimes V_Q \oplus \ldots)\big/(\ldots, -Q(v) + v \otimes v, \ldots)_{v \in V_Q}.$$

Let $e_1, \ldots, e_{2g+2}$ be the standard basis vectors of V_Q, then the $C(Q)$ is generated by $\mathbf{C}$ and the e_j with relations:

$$e_j \cdot e_j = 1, \qquad e_j \cdot e_k + e_k \cdot e_j = 0 \quad (j \neq k),$$

here $\cdot$ stands for the product induced by $\otimes$ on $C(Q)$. The (associative) algebra $C(Q)$ becomes a Lie algebra by defining, as usual, the Lie bracket to be $[x, y] := x \cdot y - y \cdot x$.

The following proposition relates the Heisenberg group, the Clifford algebra and the spin representation of the Lie algebra $so(2g+2)$.

4.4.5 Proposition.

With the notation as above we have:

1. The $\mathbf{C}$-linear map

   $$\gamma^\sharp : C(Q) \longrightarrow \mathrm{End}(V^\sharp), \qquad \lambda \mapsto \lambda I, \quad e_k \mapsto U^\sharp_{x_{k,\,2g+4}} \qquad (\lambda \in \mathbf{C}, \;\; k = 1, \ldots, 2g+2)$$

   defines an isomorphism of (associative $\mathbf{C}$-) algebras.

2. The linear map:

   $$\rho_s^\sharp : so(2g+2) \longrightarrow \mathrm{End}(V^\sharp) \cong_{\gamma^\sharp} C(Q), \qquad F_{jk} \longmapsto U^\sharp_{x_{j,2g+4}} U^\sharp_{x_{k,2g+4}} \;(= \gamma^\sharp(e_j \cdot e_k))$$

   is an injective homomorphism of Lie algebras.

3. The subspace $V_+^\sharp$ of $V^\sharp$ is invariant under the action of $so(2g+2)$. The Lie algebra representation

   $$\rho_s : so(2g+2) \longrightarrow \mathrm{End}(V_+^\sharp) \cong \mathrm{End}(V), \qquad \rho_s(x) := \rho_s^\sharp(x)\,\Big|_{V_+^\sharp}$$

   is an (irreducible) half spin representation of $so(2g+2)$. In particular, $V \cong V(\omega_{g+1})$.

4. We have

   $$\rho_s(F_{jk}) = \pm i U_{x_{jk}}, \qquad (F_{jk} \in so(2g+2), \quad i^2 = -1),$$

   where the $U_{x_{jk}}$ $(\in \mathrm{End}(V))$ with $x_{jk} \in \mathbf{F}_2^{2g}$ are in the Schrödinger representation of the Heisenberg group G.

Proof. By definition we have $(U^\sharp_{x'})^2 = I$ for all $x' \in \mathbf{F}_2^{2g+2}$. Moreover, $E^\sharp(x_{\{k,2g+4\}}, x_{\{l,2g+4\}}) = 1$ when $k \neq l$ so the corresponding maps anti-commute: $U^\sharp_{x_{k,2g+4}} U^\sharp_{x_{l,2g+4}} = -U^\sharp_{x_{l,2g+4}} U^\sharp_{x_{k,2g+4}}$. Then the map $\gamma^\sharp$ preserves the relations in $C(Q)$ and is thus an algebra homomorphism. It is surjective because the matrices $U^\sharp_{x'}$, where x' runs over $\mathbf{F}_2^{g+1} \times \mathbf{F}_2^{g+1}$ are a basis of $\mathrm{End}(V^\sharp)$ (the Schrödinger representation being irreducible) and any $U^\sharp_{x'}$ is a product (up to scalar multiple) of suitable $U^\sharp_{x_{j,2g+4}}$. Therefore $\gamma^\sharp$ is an isomorphism since both algebras have the same dimension.
(2) This is worked out in [FH], Lemma 20.7. With their notations and our Q, we have

$$\phi : \wedge^2 V_Q \stackrel{\cong}{\longrightarrow} so(2g+2), \qquad e_j \wedge e_k \longmapsto F_{jk},$$

$$\psi : \wedge^2 V_Q \longrightarrow C(Q), \qquad e_j \wedge e_k \longmapsto e_j \cdot e_k \qquad (j \neq k), \qquad \text{and} \quad \rho^\sharp_s = \psi \circ \phi^{-1}.$$

(3) By definition of $\rho^\sharp_s$ and the fact that $U^\sharp$ is a representation we have:

$$\rho^\sharp_s(F_{jk}) = U^\sharp_{x_{j,2g+4}} U^\sharp_{x_{k,2g+4}} = c_{jk} U^\sharp_{x_{jk}},$$

with $1 \leq j, k \leq 2g+2$ and some $c_{jk} \in \mathbf{C}^*$. Thus the $\rho^\sharp_s(F_{jk})$ commute with $U^\sharp_{p'}$ in $\mathrm{End}(V^\sharp)$. Therefore we obtain a Lie algebra representation on $V^\sharp_+$ cf. [FH], Prop. 20.15 (and identify their $End(\wedge^{even} W)$ with our $End(V^\sharp_+)$), where also the irreducibility is proved.
(4) We have $\rho^\sharp_s(F_{jk}) = U^\sharp_{x_{j,2g+4}} U^\sharp_{x_{k,2g+4}}$, thus $\rho^\sharp_s(F_{jk})^2 = -I$ since these two elements anti-commute. Therefore $\rho^\sharp_s(F_{jk}) = \pm i U^\sharp_{x_{jk}}$, which acts as $\pm i U_{x_{jk}}$ on $V^\sharp_+$. QED

4.5 The Hitchin connection

4.5.1

The symmetric group S_{2g+2} acts on $\mathcal{P}$ by permuting the coordinates, the quotient wil be denoted by $\overline{\mathcal{P}}$. The fundamental group of $\mathcal{P}$ is the pure braid group and $\pi_1(\overline{\mathcal{P}}) = B_{2g+2}$, the Braid group.

In case $g = 2$, we have $S_6 \cong Sp(4, \mathbf{F}_2)$, in fact there is a surjective map (which factors over the mapping class group)

$$\pi_1(\overline{\mathcal{P}}) = B_6 \longrightarrow Sp(4, \mathbf{Z}),$$

the kernel of the composition $B_6 \to Sp(4, \mathbf{Z}) \to Sp(4, \mathbf{F}_2)$ is $\pi_1(\mathcal{P})$.

¿From the theory of theta functions we know that the groupscheme $\mathcal{G}$ is trivialized on the cover $\tilde{\mathcal{P}}$ of $\mathcal{P}$ defined by the kernel of the composition:

$$\phi : \pi_1(\overline{\mathcal{P}}) \longrightarrow Sp(4, \mathbf{Z}) \longrightarrow Sp(4, \mathbf{Z})/\Gamma_2(2,4) \cong A(G).$$

Here we use Igusa's notation:

$$\Gamma_2(2,4) := \left\{ \begin{pmatrix} I + 2A & 2B \\ 2C & I + 2D \end{pmatrix} \in Sp(4, \mathbf{Z}) : \ diagonal(B) \equiv diagonal(C) \equiv (0,0) \bmod 2 \right\}.$$

Thus we consider the following diagram of etale Galois coverings:

$$\tilde{\mathcal{P}} \longrightarrow \mathcal{P} \longrightarrow \overline{\mathcal{P}} := \mathcal{P}/S_6,$$

and the corresponding exact sequence of covering groups:

$$0 \longrightarrow \mathbf{F}_2^{2g} \longrightarrow A(G) \longrightarrow S_6 \longrightarrow 0.$$

Recall that we have defined a projective representation $\tilde{T} : A(G) \to PGL(V)$ in 4.2.4. This induces projective representations on each S^kV.

4.5.2 Theorem. With the notation as in 4.5.1 (except for S_k) and $g = 2$ we have:

1. There is a line bundle $\mathcal{N}$ on $\tilde{\mathcal{P}}$ such that the pull-back of $p_*\mathcal{L}^{\otimes k}$ is isomorphic to $S_k \otimes_{\mathbf{C}} \mathcal{N}$. Here $S_k = S^kV(\omega_{g+1})$ and $V(\omega_{g+1})$ is a half spin representation of $so(6) \cong sl(4)$ (which is the standard representation (or its dual) of $sl(4)$).

2. The Hitchin connection on the pull-back of $p_*\mathcal{L}^{\otimes k}$ to $\tilde{\mathcal{P}}$ is given by the pull-back to $\tilde{\mathcal{P}}$ of the connection on $S_k \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}}$ defined by the one form

$$\frac{-1}{16(k+2)} \sum_{i,j\, i\neq j} \frac{\sigma\tilde{\rho}_s(\Omega_{ij})\mathrm{d}z_i}{z_i - z_j}$$

with $\tilde{\rho_s} : U(so(2g+2)) \to \mathrm{End}(S_1)$ the half spin representation (then the $\sigma\tilde{\rho}_s(\Omega_{ij})$ give endomorphisms of each S_k, cf. 3.2.5).

3. The Hitchin connection on the bundle $S_k \otimes \mathcal{O}_{\tilde{\mathcal{P}}}$ over $\tilde{\mathcal{P}}$ descends to a projective flat connection on the the bundle $(S_k \otimes \mathcal{O}_{\tilde{\mathcal{P}}})/A(G)$ over $\overline{\mathcal{P}}$, where $A(G)$ acts on S_k via the k-th symmetric power of its projective representation on S_1 and $A(G)$ acts on $\tilde{\mathcal{P}}$ as $Gal(\tilde{\mathcal{P}}/\overline{\mathcal{P}})$. This descent gives the kth Hitchin connection of the natural descend of the family of curves described in 1.4.

Proof. For the first part it suffices to show that the pull-back of $p : \mathbf{P} \to \mathcal{P}$ to $\tilde{\mathcal{P}}$ is trivial. Since $\mathcal{G}$ is isomorphic to the constant group scheme G (cf. 4.2.1 (with here $t \in \mathbf{C}^*$)), the uniqueness of the Schrödinger representation gives the global triviallization. We already observed that the line bundle $\mathcal{N}$ does not interfere with the projective connections.
(2) We use Theorem 2.6.9 so we must verify $(Eq.)$ from 2.6.7. The covariant derivatives given are defined by the heat operator

$$X = -\frac{1}{16(k+2)} \sum_{j\neq i} \sigma\tilde{\rho}_s(\Omega_{ij}) \otimes \frac{\mathrm{d}z_i}{z_i - z_j} \qquad (\in (S_2 \otimes_{\mathbf{C}} S_2^*) \otimes \Omega^1_{\mathcal{P}}).$$

The Heisenberg-invariance of this operator follows from the facts that $\Omega_{ij} = F_{ij} \otimes F_{ij}$, that $\rho_s(F_{kj}) = \pm iU_{x_{jk}}$ with $i^2 = -1$ and that $U_yU_xU_y^{-1} = \pm U_x$. Since $S^4(\tilde{\rho}_s)$ is a subrepresentation of $\tilde{\rho} : so(2g+2) \to \mathrm{End}S^2(\wedge^2\mathbf{C}^{2g+2})$ we have $\sigma\tilde{\rho}_s(\Omega_{ij}) = \sigma\tilde{\rho}(\Omega_{ij})$ on S_4. For the connection ∇_0 on $\tilde{p}_*\mathcal{O}_{\mathbf{P}}(1) = S_1 \otimes \mathcal{O}_{\tilde{P}}$ we simply take $\nabla_0(fw) = w \otimes \mathrm{d}f$. Then the equation $(Eq.)$ is exactly the statement of Theorem 3.4.3.
(3) We have a smooth projective family of genus 2 curves over the scheme $\bar{\mathcal{P}}$ given by

$$y^2 = f(x) = x^6 + a_1x^5 + a^2x^4 + ... + a_6 = \prod_i (x - z_i),$$

where a_i is the i-th symmetric function of the z_j. Thus by the general theory of Section 2 we have a $\mathbf{P}^3$-bundle $\bar{p} : \bar{\mathbf{P}} \longrightarrow \bar{\mathcal{P}}$ and a projective Hitchin connection $\bar{\mathbf{D}}_k$ on $\bar{p}_*\mathcal{O}(k)$. Let $\varphi : \tilde{\mathcal{P}} \longrightarrow \bar{\mathcal{P}}$ be the natural map. Then there is an isomorphism

$$(*) \qquad \varphi^*(\bar{\mathbf{P}}) = \mathbf{P}^3 \times \tilde{\mathcal{P}}$$

which is compatible with the action of $A(G)$ (which lies over the action of $A(G)$ on $\tilde{\mathcal{P}}$) on both sides: this isomorphism is given by the trivialization of the action of the theta-group scheme $\mathcal{G}$ which was mentioned earlier. The isomorphism (*) induces an isomorphism

$$(**) \qquad \varphi^*(\bar{p}_*O(k)) \cong S_k \otimes O_{\tilde{\mathcal{P}}}$$

which is compatible with the natural projective action of $A(G)$ on the LHS (via $\mathcal{G}$) and the action of $A(G)$ on the RHS via the Sym^k of its natural representation on S_1. The Heat operator given by X (as in proof of 4.5.2.2, see 2.6.7) on the LHS of (*) is VIA (*) compatible with the pullback of the Hitchin heat operator of the LHS of (*). But both heat operators are invariant under the action of $A(G)$. Thus the projective connections on the RHS and LHS of (**) agree and are compatible with the projective action of $A(G)$ on both sides. In other words the natural descent datum on the LHS of (**) agrees with the descent datum on the RHS of (**). QED

4.5.3 Examples

We give some examples of the $\rho_s(\Omega_{ij})$'s in case $g = 2$. We identify $V = S_1$, with its standard basis X_σ, $\sigma \in \mathbf{F}_2^2$. A linear map with matrix (a_{kl}) is then also given by the linear differential operator $\sum_{kl} a_{kl} X_k \partial_l$ (since $\partial_l(X_m) = 0$ for $l \neq m$ and is 1 if $l = m$). Recall from 4.3.2 that (for $g = 2$):

$$x_{12} = ((0,0),(1,0)), \qquad \text{so} \quad U_{x_{12}} = X_{00}\partial_{00} + X_{01}\partial_{01} - X_{10}\partial_{10} - X_{11}\partial_{11}.$$

Similarly, we had $x_{16} = ((1,0),(0,0))$ and $x_{26} = ((1,0),(1,0))$ and so:

$$U_{x_{16}} = X_{10}\partial_{00} + X_{11}\partial_{10} + X_{00}\partial_{10} + X_{01}\partial_{10}, \quad U_{x_{26}} = i(-X_{10}\partial_{00} - X_{11}\partial_{10} + X_{00}\partial_{10} + X_{01}\partial_{10}).$$

Since $U_{x_{jk}} = \pm i\rho_s(F_{jk})$ and $U_{x_{jk}}^2 = I$ we get (cf. 3.2.5):

$$\sigma(\tilde{\rho}_s(\Omega_{12})) = -2(X_{00}^2\partial_{00}^2 + X_{01}^2\partial_{01}^2 + X_{10}^2\partial_{10}^2 + X_{11}^2\partial_{11}^2 + 2X_{00}X_{01}\partial_{00}\partial_{01} - 2X_{10}X_{11}\partial_{10}\partial_{11}),$$

4.6 Local Monodromy

4.6.1 Introduction

We want to obtain some information on the representation of the mapping class group for genus two curves defined by Hitchin's connection. This representation has been studied, for arbitrary g and k, by Moore and Seiberg [MS], Kohno [K2] and in the case $k = 2$ by Wright [Wr]. In some sense, we merely find (weaker, local) results which do agree with their results.

4.6.2 The method

We recall how to determine the local monodromy on the trival bundle $\mathcal{O}_{\mathcal{P}} \otimes_{\mathbf{C}} W$ with a connection ∇. We consider a holomorphic map

$$\phi : D^* \longrightarrow \mathcal{P}, \qquad \text{with} \quad D^* := \{t \in \mathbf{C} - \{0\} : \ |t| < \epsilon\}.$$

(for small, positive ϵ) and pull-back the one form $\omega \in \mathrm{End}(W) \otimes \Omega^1_{\mathcal{P}}$ which defines the connection (so $\nabla(fw) = w\mathrm{d}f - f\omega(w)$. We write

$$\phi^*\omega = \left(\frac{R}{t} + A(t)\right) \mathrm{d}t, \qquad R \in \mathrm{End}(W)$$

with $A(t)$ holomorphic for $t = 0$, and R is called the residue at $t = 0$. We identify $\pi_1(D^*) = \mathbf{Z}$; $1 \in \mathbf{Z}$ represents a small circle traversed in anti-clockwise direction. The eigenvalues of the monodromy of $1 \in \pi_1(D^*)$ are the $exp(-2\pi i \mu_j)$ were the μ_i are the eigenvalues of R on W.

In case the monodromy transformations are semi-simple, the local monodromy is conjugated to $exp(-2\pi i R)$ (cf. [D], p. 54). However we could not prove the semi-simplicity (but it seems to be known to the physicists).

To get the eigenvalues of the monodromy of the Hitchin connection for a $\gamma \in \pi_1(\overline{\mathcal{P}})$, let n be the order of its image in $Gal(\mathcal{P}/\overline{\mathcal{P}})$ and let $\bar{\gamma}$ be its image in $A(G) = Gal(\tilde{\mathcal{P}}/\overline{\mathcal{P}})$. We choose a $\phi : D^* \to \mathcal{P}$ in such a way that $\phi_*(1) = \gamma^n$ $(\in \pi_1(\mathcal{P}))$. Then these eigenvalues are the eigenvalues of the matrix $exp(\frac{-2\pi i}{n} R)\tilde{T}_{\bar{\gamma}}$, with R determined as above for the corresponding connection on $\mathcal{P}$ (cf. [K1]). Since $\tilde{T}$ is a projective representation (on S_1 and thus on any S_k), the set of eigenvalues are only defined up to multiplication by one non-zero constant. This corresponds to the fact that we only have a projectively flat connection.

4.6.3 Non-seperating vanishing cycle

Let $\gamma \in \pi_1(\bar{\mathcal{P}})$ such that $\gamma^2 = \phi_*(1)$ with

$$\phi : D^* \longrightarrow \mathcal{P}, \qquad t \longmapsto (t + z_2, z_2, \ldots, z_6)$$

where we fix distinct $z_j \in \mathbf{C}$. (Then γ corresponds to a Dehn twist in a simple non-seperating loop in the mapping class group.) The residue of $\phi^*\omega$ for the connection on $S_k \otimes \mathcal{O}_{\mathcal{P}}$ corresponding to the Hitchin connection on $p_*\mathcal{L}^{\otimes k}$ is (cf. 4.5.2 and 4.5.3): $\frac{-1}{16(k+2)}\sigma\tilde{\rho}_s(\Omega_{12})$,

$$-\tilde{\rho}_s(\Omega_{12}) = R_{12} := \Big((X_{(0,0)}\partial_{(0,0)} + X_{(0,1)}\partial_{(0,1)}) - (X_{(1,0)}\partial_{(1,0)} + X_{(1,1)}\partial_{(1,1)})\Big)^2$$

4.6.4 Lemma. The eigenvalues of of the monodromy of γ as in 4.6.3 on S_k are:

$$\lambda_c := exp(-2\pi i \tfrac{c(c+1)}{k+2}), \qquad \mathrm{mult}(\lambda_c) = (k - 2c + 1)(2c + 1)$$

(up to multiplication by a non-zero constant independent of c), with $2c \in \mathbf{Z}$ and $0 \le c \le \frac{k}{2}$.

Proof. Since the eigenvalues are only determined up to a constant and since the difference between $2R_{12}$ and $\sigma(R_{12})$ is a multiple of the Euler vector field, which acts by multiplication by k on S_k, it suffices to consider the eigenvalues of R_{12}. For a monomial in S_k we have:

$$R_{12}(X^{l_0}_{(0,0)}X^{l_1}_{(0,1)}X^{m_0}_{(1,0)}X^{m_1}_{(1,1)}) == ((l_0+l_1)-(m_0+m_1))^2(X^{l_0}_{(0,0)}X^{l_1}_{(0,1)}X^{m_0}_{(1,0)}X^{m_1}_{(1,1)})$$

Thus each monomial is an eigenvector. Let $b := m_0+m_1$. Since the monomial has degree k, its eigenvalue is $(k-2b)^2$. The multiplicity of the eigenvalue is $(k-b+1)(b+1)$ (the dimension of the space of homogeneous polynomials in 2 variables of degree c is $c+1$). Thus the eigenvalue $2(k-2b)^2$ of $2R_{12}$ has multiplicity $(k-b+1)(b+1)$.

The image $\bar{\gamma}$ of γ in $A(G)$ is $\tilde{T}_{x_{12}}$ which acts on such a monomial by (cf. 4.2.7):

$$\tilde{T}_{x_{12}}(X^{l_0}_{(0,0)}X^{l_1}_{(0,1)}X^{m_0}_{(1,0)}X^{m_1}_{(1,1)}) = i^b(X^{l_0}_{(0,0)}X^{l_1}_{(0,1)}X^{m_0}_{(1,0)}X^{m_1}_{(1,1)}).$$

The eigenvalues of γ on S_k are then, with $c := b/2$:

$$\begin{aligned} exp(\tfrac{-2\pi i}{2}\left(\tfrac{+1}{16(k+2)}2(k-2b)^2+\tfrac{b}{4}\right)) &= exp(-2\pi i\left(\tfrac{k^2-8ck+16c^2+8ck+16c}{16(k+2)}\right)) \\ &= exp(-2\pi i\left(\tfrac{k^2}{16(k+2)}\right))\, exp(-2\pi i\left(\tfrac{c(c+1)}{k+2}\right)). \end{aligned}$$

QED

4.6.5 Seperating vanishing cycle

Now we consider a $\gamma\in\pi_1(\bar{\mathcal{P}})$ such that $\gamma=\phi_*(1)$ with

$$\phi: D^*\longrightarrow\mathcal{P},\qquad t\longmapsto(tz_1,tz_2,tz_3,z_4,z_5,z_6),$$

with distinct nonzero z_i's.

The residue in $t=0$ of the connection on $S_k\otimes\mathcal{P}$ corresponding to Hitchin's connection is

$$\tfrac{-1}{16(k+2)}R_{123},\qquad\text{with}\quad R_{123}:=2\sigma(\tilde{\rho}_s(\Omega_{12}+\Omega_{13}+\Omega_{23})).$$

4.6.6 Lemma. Let $g=2$.

1. There are constants $\lambda_k\in\mathbf{Q}$ such that (in $\mathrm{End}(S_k)$):

$$R_{123}=16QX_Q+\lambda_k I\quad\text{with}\quad QX_Q=(X_{00}X_{10}-X_{01}X_{11})(\partial_{00}\partial_{10}-\partial_{01}\partial_{11}).$$

2. We define a subspace of S_k by:

$$V_k := Kernel(X_Q:=\partial_0\partial_1-\partial_2\partial_3:\, S_k\longrightarrow S_{k-2}).$$

   Then the vector space S_k is a direct sum:

$$S_k=V_k\oplus QV_{k-2}\oplus Q^2V_{k-4}\oplus\ldots$$

   Moreover, each subspace Q^lV_{k-2l} is an eigenspace of R_Q with eigenvalue:

$$\lambda_l=l(k-l+1)\qquad\text{and}\qquad\dim V_{k-2l}=(k-2l+1)^2.$$

3. The eigenvalues of of the monodromy of γ on S_k are
$$\lambda_l := exp(-2\pi i \tfrac{l(l+1)}{k+2}) \qquad mult(\lambda_l) = (k-2l+1)^2$$
(up to multiplication by a constant independent of l), with $l \in \mathbf{Z}$ and $0 \leq l \leq k/2$.

Proof. Recall x_{12} corresponds to $((0,0),(1,0)) \in \mathbf{F}_2^2 \times \mathbf{F}_2^2$ and $x_{23} = x_{\{2,3,4,5\}} + x_{\{4,5\}}$ corresponds to $((1,1),(0,0))$ thus $x_{13} = x_{12} + x_{23}$ corresponds to $((1,1),(1,0))$. Since $(1,1)(1,0) = 1$, we have a $+$ sign in front of Ω_{13} below. Let $\tilde{\rho_s}(\Omega_{00})$ be the square of the Euler vector field (which acts as $k^2 I$ on S_k). Then:

$$\begin{aligned}
\tilde{\rho_s}(\Omega_{00}) &= +(X_{00}\partial_{00} + X_{01}\partial_{01} + X_{10}\partial_{10} + X_{11}\partial_{11})^2, \\
\tilde{\rho_s}(\Omega_{12}) &= -(X_{00}\partial_{00} + X_{01}\partial_{01} - X_{10}\partial_{10} - X_{11}\partial_{11})^2, \\
\tilde{\rho_s}(\Omega_{23}) &= -(X_{11}\partial_{00} + X_{10}\partial_{01} + X_{01}\partial_{10} + X_{00}\partial_{11})^2, \\
\tilde{\rho_s}(\Omega_{13}) &= +(X_{11}\partial_{00} + X_{10}\partial_{01} - X_{01}\partial_{10} - X_{00}\partial_{11})^2.
\end{aligned}$$

The sum of these operators, minus the (degree one) parts which act as constants on each S_k, is $4QX_Q$. Remembering the factor 2 in R_{123} and in our symbols, we find the first statement.
(2) This is well-known. Let $f \in V_n$, so f is homogeneous of degree n and $X_Q(f) = 0$. Then an elementary computation gives for $l \geq 1$ (with variables X_i and $Q = X_0X_1 - X_2X_3$):

$$\begin{aligned}
& (\partial_0\partial_1 - \partial_2\partial_3)(fQ^l) \\
= \;& (\partial_0 f)(\partial_1 Q^l) + (\partial_0 Q^l)(\partial_1 f) - (\partial_2 f)(\partial_3 Q^l) - (\partial_2 Q^l)(\partial_3 f) + f(\partial_0\partial_1 - \partial_2\partial_3)(Q^l) \\
= \;& lQ^{l-1}(X_0\partial_0 + X_1\partial_1 + X_2\partial_2 + X_3\partial_3)(f) + l(l+1)fQ^{l-1} \\
= \;& l(n+l+1)fQ^{l-1}.
\end{aligned}$$

Writing $n = k - 2l$ we get, for all integers l with $0 \leq 2l \leq k$:
$$QX_Q(fQ^l) = l(k-l+1))fQ^l, \qquad f \in V_{k-2l}.$$
Hence each $V_{k-2l}Q^l$ is an eigenspace for R_Q. By induction (the cases $k = 0, 1$ being trivial) we may assume that $S_{k-2} = \oplus S_{k-2-2l}Q^l$. Therefore $QS_{k-2} = \oplus Q^{l+1}S_{k-2(l+1)} \subset S_k$ is a direct sum of eigenspaces of R_Q and none of the eigenvalues of R_Q on this subspace is zero ($k + l(l+1) > 0$ for $k \geq 2$, $l \geq 0$). Thus $\ker(R_Q) \cap QS_{k-2} = \{0\}$ and R_Q induces an isomorphism on QS_{k-2}. Therefore $S_k = Ker(R_Q) \oplus Im(R_Q) = \oplus Q^l S_{k-2l}$. The dimension of V_k is then $\dim S_k - \dim S_{k-2} = \binom{k+3}{3} - \binom{k+1}{3} = (k+1)^2$.
(3) The image of γ in $A(G)$ is trivial. The eigenvalues of $R_{123} = 8QX_Q$ on S_k are $16l(k-l+1)$. Thus the eigenvalues of γ are (see 4.6.2):

$$\begin{aligned}
exp(-2\pi i \tfrac{-1}{16(k+2)} 16l(k-l+1)) &= exp(-2\pi i \left(\tfrac{l^2-l-l(k+2)+2l}{k+2}\right)) \\
&= exp(-2\pi i \tfrac{l(l+1)}{k+2}).
\end{aligned}$$

QED

4.6.7 Kohno's results

The monodromy of the Hitchin connection has been studied by Moore-Seiberg and Kohno (among others), we will relate the results above to those contained in [K2] for $g = 2$.

Let γ be a tri-valent graph with two edges, so there are three edges meeting in each vertex. For $k \in \mathbf{Z}_{\geq 0}$ we define a finite set $b_k(\gamma)$ of functions on the edges of γ by:

$$f : Edges(\gamma) \longrightarrow \{0, \tfrac{1}{2}, 1, \dots, \tfrac{k}{2}\} \quad \text{with} \quad \begin{cases} f(e_i) + f(e_j) + f(e_k) \leq k, \\ |f(e_i) - f(e_j)| \leq f(e_k) \leq f(e_i) + f(e_j), \\ f(e_i) + f(e_j) + f(e_k) \in \mathbf{Z}, \end{cases}$$

for any three edges e_i, e_j, e_k meeting in a vertex. The Verlinde space V_k is the $\mathbf{C}$-vector space with basis $b_k(\gamma)$, it has the same dimension as S_k.

The graph γ corresponds to a pants decomposition of a genus two Riemann surface. Each vertex is a pant, homeomorphic to $\mathbf{P}^1$ minus 3 points, the edges correspond to these points, which in turn correspond to 'vanishing cycles' on the Riemann surface. The mapping class group has a projective representation on V_k. The Dehn twist in a vanishing cycle corresponding to an edge e of γ acts, up to a scalar multiple, conjugated to the diagonal matrix with entries (cf. [K2], p. 217; p. 214, (2-1))

$$exp(-2\pi i f(e)(f(e)+1)/(k+2) \qquad (f \in b_k(\gamma)).$$

Using this recipe we find the same results as before (see Lemmas 4.6.4 and 4.6.6):

4.6.8 Lemma. With the notation as above:

1. The eigenvalues of the Dehn twist in a non-seperating cycle on V_k are $exp(-2\pi i v(v+1)/(k+2))$ with multiplicity $(k-2v+1)(2v+1)$ where $v \in \{0, \frac{1}{2}, 1, \dots, \frac{k}{2}\}$.
2. The eigenvalues of a Dehn twist in a seperating cycle on V_k are $exp(-2\pi i l(l+1)/(k+2))$ with multiplicity $(k-2l+1)^2$ where l is an integer with $0 \leq l \leq k/2$.

Proof. Consider the graph γ with two vertices joined by three edges e_1, e_2, e_3. We first determine the set $b_k(\gamma)$. The edge e_1 corresponds to a non-seperating cycle and we determine $f(e_1)$ for all $f \in b_k(\gamma)$, this will give the first result.

Writing $f_i := f(e_i)$ for $f \in b_k(\gamma)$, the conditions are the same in each of the two edges. Assume that we have $f_1 \leq f_2 \leq f_3$. The conditions on the f_i are then equivalent to the following 3 conditions:

$$f_1 + f_2 + f_3 \leq k, \qquad f_3 \leq f_1 + f_2, \qquad f_1 + f_2 + f_3 \in \mathbf{Z}, \qquad f_i \in \{0, \tfrac{1}{2}, 1, \dots, \tfrac{k}{2}\}$$

In case $f_1 + f_2 + f_3 = l$ $(\leq k)$ and $f_3 > l/2$, the second condition implies $f_1 + f_2 > l/2$, a contradiction. Thus the set $b_k(\gamma)$ is a disjoint union of sets $c_l(\gamma)$ for $l \in \mathbf{Z}$ and $0 \leq l \leq k$:

$$c_l(\gamma) := \{f \in b_k(\gamma) : \; f_1 + f_2 + f_3 = l, \quad f_i \leq l/2\, \}, \qquad b_k(\gamma) = \dot{\cup}_{0 \leq l \leq k}\, c_l(\gamma).$$

Next we observe that if $f \in c_l(\gamma)$ then $f_1+f_2+f_3=l$ implies $f_1+f_2=l-f_3 \geq l/2$ (since $f_3 \leq l/2$), thus certainly $f_3 \leq f_1+f_2$, so the second condition is always fulfilled.

It remains to determine the number of triples (f_1, f_2, f_3) with $f_i \in \{0, \frac{1}{2}, 1, \ldots, \frac{l}{2}\}$ and $f_1+f_2+f_3=l$. Equivalently, we have to find the number of monomials $X^aY^bZ^c$ with $a+b+c=2l$ and $0 \leq a,\, b,\, c \leq l$. If such a monimial with $a+b+c=2l$ does not satifisfy $0 \leq a,\, b,\, c \leq l$, then at most one of a, b, c can be $> l$, say c, and then $a+b=2l-c$ which gives $(2l-c)+1$ possible couples a, b. Varying c between $l+1$ and $2l$ we get $l+(l-1)+\ldots+1=(1/2)l(l+1)$ monomials with $c>l$ and thus there are $(3/2)l(l+1)$ monomials of degree $2l$ which do not satisfy the condition $0 \leq a,\, b,\, c \leq l$. Then

$$|c_l(\gamma)| = \binom{2l+2}{2} - (3/2)l(l+1) = \binom{l+2}{2}, \quad \text{hence} \quad |b_k(\gamma)| = \sum_{l=0}^{k} \binom{l+2}{2} = \binom{k+3}{3},$$

so indeed $|b_k(\gamma)| = \dim S_k$.

Next we determine the number of triples $(v, f_2, f_3) \in b_k(\gamma) = \dot\cup\, c_l(\gamma)$. First we will count such triples in $c_l(\gamma)$. Let $a := 2v \in \mathbf{Z}_{\geq 0}$ with $a \leq l$, then there are $2l-a+1$ couples (b,c) with $a+b+c=2l$. Of these, only $a+1$ have $b,\, c \leq l$ (they are $(l-a, l)$, $(l-a+1, l-1), \ldots, (l, l-a)$). This gives $a+1$ such triples in $c_l(\gamma)$ provided $f_1 = a/2 \leq l/2$. Thus v occurs for $l/2 = v,\, v+1/2, \ldots, k/2$, that is for $k-2v+1$ values of $l/2$. Thus there are $(k-2v+1)(2v+1)$ triples (v, f_2, f_3).

(2) We now consider the case of a seperating vanishing cycle. The recipe we folow is not explicitly given in [K2], but is similar to the one used above and gives a result that agrees with Lemma 4.6.6.

Let δ be the graph with two vertices v_1, v_2 and three edges e_1, e_2, e_3 such that begin and end of e_i is v_i and where e_3 connects v_1 and v_2. Thus the edge e_3 corresponds to a seperating vanishing cycle.

As before, let $f_i := f(e_i)$. The second set of inequalities on the $f(e_i)$ reduce to

$$f_3 \leq 2f_1, \qquad f_3 \leq 2f_2; \qquad \text{moreover} \quad f_1+f_1+f_3 \in \mathbf{Z} \Rightarrow f_3 \in \mathbf{Z} \cap \{0, 1/2, \ldots, k/2\},$$

this is the third condition for the vertex v_1. Finally we have $2f_i + f_3 \leq k$ so $f_3/2 \leq f_i \leq (k-f_3)/2$ for $i=1,\,2$. Thus given f_3, we find $(k-2f_3)+1)^2$ possibilities for couples f_1, f_2 ($f_i \in \{\frac{f_3}{2}, \frac{f_3+1}{2}, \ldots, \frac{k-f_3}{2}\}$).

We observe that for fixed f_1, f_3 may have the values $0, 1, \ldots, 2f_1$, correspondingly f_2 has $k+1,\, k+1-2, \ldots, k+1-4f_1$ different values. Thus the multiplicity of f_1 is, again, $(2f_1+1)((k+1)-2(1+2+\ldots+2f_1)=(2f_1+1)(k+1-2f_1)$. QED

4.6.9 The cases $k = 1,\, 2$

The case $k=1$ is particularly easy since the operators Ω_{ij} are homogeneous of degree two and thus act as zero on S_1. The monodromy representation then factors over the projective representation of $A(G)$.

In case $k=2$, the space S_k is a direct sum of 10 distinct, one dimensional G-representations (this is easy to verify, see also [vG]). On each eigenspace, any $\sigma\tilde{\rho}_s(\Omega_{ij})$ acts as scalar multiplication by an integer. A (local) basis of flat sections of $S_2 \otimes \mathcal{P}$ is then given by 10 functions of the type $(\prod(z_i - z_j)^{r_{ijm}})Q_m$ with $r_{ij} \in \mathbf{Q}$ and $Q_m \in S_2$ is a basis of the eigenspace. These eigenspaces are permuted by the projective representation of $A(G)$.

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