KUGA-SATAKE VARIETIES AND THE HODGE CONJECTURE

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Introduction

Kuga-Satake varieties are abelian varieties associated to certain weight two Hodge structures, for example the second cohomology group of a K3 surface. We start with an introduction to Hodge structures and we give a detailed account of the construction of Kuga-Satake varieties. The Hodge conjecture is discussed in section 2. An excellent survey of the Hodge conjecture for abelian varieties is [G].

We point out a connection between the Hodge conjecture for abelian varieties and Kuga-Satake varieties in section 9. In section 10 we discuss the implications of the Hodge conjecture on the geometry of surfaces whose second cohomology group has a Kuga-Satake variety. We conclude with some recent results, inspired by an example of C. Voisin, on Kuga-Satake varieties of Hodge structures on which an imaginary quadratic field acts.

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1. Polarized Hodge structures

1.1. **Definition.** A (rational) Hodge structure of weight k ($\in \mathbf{Z}$) is a $\mathbf{Q}$-vector space V with a decomposition of its complexification $V_{\mathbf{C}} := V \otimes_{\mathbf{Q}} \mathbf{C}$:

$$V_{\mathbf{C}} = \oplus_{p+q=k} V^{p,q}, \qquad \text{and} \quad \overline{V^{p,q}} = V^{q,p} \qquad (p,\, q \in \mathbf{Z}).$$

Here complex conjugation on $V_{\mathbf{C}}$ is given by $\overline{v \otimes z} := v \otimes \bar{z}$ for $v \in V$ and $z \in \mathbf{C}$. We will take $k,\, p,\, q \geq 0$ except in some of the proofs.

1.2. The (Betti) cohomology groups $H^k(X, \mathbf{Q})$ of a complex smooth projective variety are Hodge structures:

$$H^k(X, \mathbf{C}) = \oplus_{p+q=k} H^{p,q}(X).$$

Identifying $H^k(X, \mathbf{C})$ with harmonic differential forms, the subspaces $H^{p,q}(X)$ consist of the harmonic forms of type (p,q) and $H^{p,q}(X) \cong H^q(X, \Omega^p)$.

1.3. It is useful to identify Hodge structures on V with certain representations of the group $\mathbf{C}^*$ on $V_{\mathbf{R}} := V \otimes_{\mathbf{Q}} \mathbf{R}$. We identify $\mathbf{C}^*$ with a subgroup of $GL(2, \mathbf{R})$:

$$\mathbf{C}^* \cong \left\{ s(a,b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2, \mathbf{R}) :\ a^2 + b^2 \neq 0 \right\}, \qquad z = a + bi \longmapsto s(a,b).$$

The eigenvalues of $s(a,b)$ are $z = a + bi$ and $\bar{z} = a - bi$ with corresponding eigenvectors $e_1 + ie_2$ and $e_1 - ie_2$, here e_j is the standard j-th basis vector of $\mathbf{C}^2$. An algebraic representation of $\mathbf{C}^*$ is defined to be a homomorphism

$$h : \mathbf{C}^* \longrightarrow GL(V_{\mathbf{R}})$$

such that, with respect to some basis of $V_{\mathbf{R}}$, the entries of the matrix $h(s(a,b))$ are polynomials, with coefficients in $\mathbf{R}$, in a, b, $(a^2+b^2)^{-1}$. The following proposition is well-known:

1.4. **Proposition.** There is a bijection between rational Hodge structures of weight k on a $\mathbf{Q}$-vector space V and algebraic representations $h:\mathbf{C}^*\to GL(V_{\mathbf{R}})$ with $h(t)=t^k$ for $t\in\mathbf{R}$. The Hodge structure defined by h is denoted by (V,h) and its Hodge decomposition is:

$$V^{p,q}:=\{v\in V_{\mathbf{C}}:\ h(z)v=z^p\bar{z}^q v\}.$$

Proof. Composing h with the inclusion $GL(V_{\mathbf{R}})\subset GL(V_{\mathbf{C}})$ we get a representation of $\mathbf{C}^*$ on $V_{\mathbf{C}}$ and the matrix coefficients of this representation are polynomials in z, $\bar{z}$ and $(z\bar{z})^{-1}$. There is a basis $\{v_i\}$ of $V_{\mathbf{C}}$ of simultaneous eigenvectors: $h(z)v_i=\lambda_i(z)v_i$ for some homomorphisms $\lambda_i:\mathbf{C}^*\to\mathbf{C}^*$. As λ_i is a polynomial in z, $\bar{z}$ and $(z\bar{z})^{-1}$ we get $\lambda_i(z)=z^p\bar{z}^q$ for some p, $q\in\mathbf{Z}$. Since also the conjugate of an eigenvalue is an eigenvalue (on the conjugated eigenvector) we have a Hodge structure.

Conversely, any element in $V_{\mathbf{C}}$ can be written as $v\otimes 1+w\otimes i$ with v, $w\in V_{\mathbf{R}}$. If $v\otimes 1+w\otimes i\in V^{p,q}$ then $v\otimes 1-w\otimes i\in V^{q,p}$. Let $\{v_r+iw_r\}_r$ be a basis of $V^{p,q}$ with $p\geq q$ and define $V_p=\langle v_r,\,w_r\rangle_r$ $(\subset V_{\mathbf{R}})$ be the span of the v_r, w_r's. Then

$$V_{\mathbf{R}}=\oplus_{p\geq q}V_p,\qquad\text{and}\quad V_p\otimes_{\mathbf{R}}\mathbf{C}=V^{p,q}\oplus V^{q,p}.$$

The representation h of $\mathbf{C}^*$ on $V_{\mathbf{R}}$ is constructed on each of the subspaces V_p.

In case $p=q$ we have $V^{p,p}=\overline{V^{p,p}}$, so $V^{p,p}=V_p\otimes\mathbf{C}$. Define $h(a+bi)v:=(a^2+b^2)^p v$ for all $v\in V_p$.

Next fix p,q with $p=q+l$ and $l>0$ and let $\{v_r+iw_r\}_r$ be a basis for $V^{p,q}$. Then the vectors $\{v_r,\,w_r\}_r\in V_{\mathbf{R}}$, are independent over $\mathbf{R}$ and $V_p=\oplus_r\langle v_r,\,w_r\rangle$. For each r define a representation of $\mathbf{C}^*$ on $\langle v_r,\,w_r\rangle\subseteq V_p$ by

$$h(a+bi)=(a^2+b^2)^q\cdot\begin{pmatrix}a&b\\-b&a\end{pmatrix}^l.$$

The eigenspaces in $V_{\mathbf{C}}$ of this representation are the $V^{p,q}$'s. $\square$

1.5. **Tensor products of Hodge structures.** We transfer the usual algebra constructions on representations to Hodge structures. Given rational Hodge structures (V,h_V), (W,h_W) of weight k_V, k_W one defines a rational Hodge structure $(V\otimes W,h_V\otimes h_W)$ of weight k_V+k_W by:

$$h_V\otimes h_W:\mathbf{C}^*\longrightarrow GL((V\otimes W)_{\mathbf{R}}),\qquad z\longmapsto[v\otimes w\mapsto(h_V(z)w)\otimes(h_W(z)w)].$$

The dual vector space $V^*:=Hom_{\mathbf{Q}}(V,\mathbf{Q})$ is also a Hodge structure (of weight $-k_V$):

$$h_V^*:\mathbf{C}^*\longrightarrow GL(V_{\mathbf{R}}^*),\qquad(h_V^*(z)f)(v):=f(h_V(z)^{-1}v),$$

here $f\in V_{\mathbf{R}}^*=Hom_{\mathbf{R}}(V_{\mathbf{R}},\mathbf{R})$ and $v\in V_{\mathbf{R}}$. The Tate Hodge structure $\mathbf{Q}(n)$ $(n\in\mathbf{Z})$ is defined by the vector space $\mathbf{Q}$ and the homomorphism:

$$h_n:\mathbf{C}^*\longrightarrow GL_1(\mathbf{R}),\qquad z\longmapsto(z\bar{z})^{-n},$$

it has weight $-2n$ and $\mathbf{Q}(n)^{p,q}=0$ unless $p=q=-n$ in which case $\mathbf{Q}(n)^{-n,-n}=\mathbf{C}$. We write $V(n):=V\otimes\mathbf{Q}(n)$, it is a Hodge structure of weight k_V-2n with $V(n)^{p,q}=V^{p+n,q+n}$.

1.6. **Morphisms of Hodge structures.** A morphism of Hodge structures $f : (V, h_V) \rightarrow (W, h_W)$ is a linear map $f : V \rightarrow W$ such that f intertwines the representations h_V and h_W up to a Tate twist:

$$f(h_V(z)v) = (z\bar{z})^n h_W(z) f(v)$$

(f gives a 'strict' morphism of Hodge structures $f : V \rightarrow W(-n)$). A morphism of Hodge structures satisfies $f_{\mathbf{C}}(V^{p,q}) \subset W^{p+n,q+n}$, here $f_{\mathbf{C}}$ is the $\mathbf{C}$-linear extension of f. Since f commutes with the $\mathbf{C}^*$-representations, both kernel and image of f are (sub) Hodge structures.

We denote by $Hom_{Hod}(V, W)$ ($\subset Hom_{\mathbf{Q}}(V, W)$) the $\mathbf{Q}$-vector space of morphisms of Hodge structures.

1.7. **Definition.** Let V be a rational Hodge structure of weight k and let $h : \mathbf{C}^* \rightarrow GL(V_{\mathbf{R}})$ be the corresponding representation. A polarization on V is a bilinear map:

$$\Psi : V \times V \longrightarrow \mathbf{Q}$$

satisfying (for all $v, w \in V_{\mathbf{R}}$):

$$\Psi(h(z)v, h(z)w) = (z\bar{z})^k \Psi(v, w)$$

and

$$\Psi(v, h(i)w) \quad \text{is a symmetric and positive definite form:}$$

$\Psi(v, h(i)w) = \Psi(w, h(i)v)$ for all $v, w \in V_{\mathbf{R}}$ and $\Psi(v, h(i)v) > 0$ for all $v \in V_{\mathbf{R}} - \{0\}$.

The map $h(i) : V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ is called the Weil operator. A polarization is a strict morphism of Hodge structures $V \otimes V \rightarrow \mathbf{Q}(-k)$.

1.8. **Lemma.** Let (V, Ψ) be a rational polarized Hodge structure of weight k. We denote the $\mathbf{C}$-linear extension of Ψ by $\Psi_{\mathbf{C}}$. Then:

1. Ψ is symmetric if k is even, and is alternating if k is odd.
2. For $x_{p,q} \in V^{p,q}$ and $y_{r,s} \in V^{r,s}$:

   $$\Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}) = 0 \qquad \text{if} \quad (p, q) \neq (s, r).$$

   In particular, the direct sum decomposition:

   $$V_{\mathbf{R}} = \oplus_p V_p, \qquad V_p \otimes_{\mathbf{R}} \mathbf{C} = V^{p,q} \oplus V^{q,p}$$

   is an orthogonal direct sum w.r.t. Q.
3. The (restriction of the) $\mathbf{C}$-bilinear form

   $$\Psi_{\mathbf{C}} : V^{p,q} \times V^{q,p} \longrightarrow \mathbf{C}$$

   is non-degenerate (so $V^{p,q} \cong_{\Psi_{\mathbf{C}}} (V^{q,p})^{dual}$).
4. If the weight is even, the quadratic form defined by the $\mathbf{R}$-linear extension of Ψ:

   $$Q : V_{\mathbf{R}} \longrightarrow \mathbf{R}, \qquad Q(v) := \Psi_{\mathbf{R}}(v, v),$$

   satisfies: $(-1)^{l-p} Q_{|V_p} > 0$ where $k = 2l$.

Proof. Note that $h(i)^2 v = h(-1)v = (-1)^k v$ for $v \in V_{\mathbf{R}}$. Now use:

$$\Psi(v, w) = \Psi((h(i)v), h(i)w) = \Psi(w, h(i)^2 v) = (-1)^k \Psi(w, v).$$

Next we observe that for all $z \in \mathbf{C}^*$:

$$(z\bar{z})^k \Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}) = \Psi_{\mathbf{C}}(h(z)x_{p,q}, h(z)y_{r,s}) = \Psi_{\mathbf{C}}(z^p\bar{z}^q x_{p,q}, z^r\bar{z}^s y_{r,s}) = z^{p+r}\bar{z}^{q+s}\Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}).$$

Hence $\Psi_{\mathbf{C}}(x_{p,q}, y_{r,s})$ can be non-trivial only when $p+r = q+s = k$. Since also $p+q = r+s = k$ the second statement follows.

Let $x \in V^{p,p}$, then $x = \lambda v$ with $v \in V_p$, $\lambda \in \mathbf{C}$ and $h(i)v = v$. Then $\Psi_{\mathbf{C}}(x,x) = \lambda^2\Psi(v,v) = \lambda^2\Psi(v, h(i)v)$ and $\Psi(v, h(i)v)$ is non-zero if $v \neq 0$. If $p \neq q$ then for any non-zero $x \in V^{p,q}$, $\bar{x} \in V^{q,p}$ and thus $x + \bar{x} \in V_p$ is non-zero. Therefore:

$$\begin{aligned} 0 &< \Psi(x+\bar{x}, h(i)(x+\bar{x})) \\ &= \Psi(x, h(i)\bar{x}) + \Psi(\bar{x}, h(i)x) + 0 + 0 \\ &= 2i^{q-p}\Psi(x,\bar{x}), \end{aligned}$$

in the last step we used the symmetry of $\Psi(x, h(i)y)$. Thus Ψ gives a non-degenerate pairing.

Since $p+q = k = 2l$, $h(i) = i^{p-q} = i^{2p-2l} = (-1)^{p-l}$ on $V^{p,q}$, and also on $V^{q,p}$, hence $h(i)$ acts as $(-1)^{l-p}$ on V_p. As $\Psi(v, h(i)v)$ is positive definite, Q is positive definite on V_p if $l-p$ is even and negative definite otherwise. $\square$

1.9. **Example.** The cohomology groups $H^k(X, \mathbf{Q})$ (as in 1.2) have a polarization see [W], Théoréme IV.7 and corollaire or [GH], p. 123.

2. The Hodge conjecture

2.1. **Hodge cycles.** The space of Hodge classes in a rational Hodge structure V of weight k is the $\mathbf{Q}$-subvector space:

$$B(V) := \begin{cases} 0 & \text{if} \quad k \text{ is odd,} \\ V \cap V^{p,p} & \text{if} \quad k = 2p. \end{cases}$$

Note that $V \hookrightarrow V_{\mathbf{C}} = V \otimes_{\mathbf{Q}} \mathbf{C}$, $v \mapsto v \otimes 1$ and the intersection $V \cap V^{p,p}$ takes place in $V_{\mathbf{C}}$.

2.2. **Algebraic cycles.** Let X be smooth algebraic variety and let Z be (any) irreducible subvariety of codimension p in X. Then Z defines a cohomology class $[Z] \in H^{2p}(X, \mathbf{Q})$. This defines a cycle class map from the Chow group (with coefficients in $\mathbf{Q}$) of codimension p cycles

$$[.] : CH^p(X)_{\mathbf{Q}} \longrightarrow H^{2p}(X, \mathbf{Q}), \qquad \sum a_i Z_i \longmapsto \sum a_i [Z_i].$$

It is well-known that the image of this map lies in the space of Hodge cycles:

$$[CH^p(X)_{\mathbf{Q}}] \subset B(H^{2p}(X, \mathbf{Q})) = H^{2p}(X, \mathbf{Q}) \cap H^{p,p}(X).$$

2.3. **The Hodge conjecture.** The Hodge conjecture asserts that:

$$[CH^p(X)_{\mathbf{Q}}] = B(H^{2p}(X, \mathbf{Q}))$$

that is, any cohomology class in $H^{2p}(X, \mathbf{Q})$ which is of type (p,p) is the class of a codimension p cycle.

2.4. The Hodge conjecture is known to be true in case $p = 1$ (it follows from an analysis of the exponential sequence) and thus is also true if $p = -1 + \dim X$ and, obviously also in the cases $p = 0, \dim X$. In other cases however the conjecture is very much open.

2.5. **Morphisms of Hodge structures and Hodge classes.** Let $f : V \to W$ be a (strict) morphism of Hodge structures. Then $f \in Hom_{\mathbf{Q}}(V, W) = V^* \otimes W$, which is a Hodge structure. Moreover, we have, for all $z \in \mathbf{C}^*$ and all $v \in V_{\mathbf{R}}$:

$$f(h_V(z)v) = h_W(z)f(v) \qquad \text{so} \quad f(v) = h_W(z)f(h_V(z)^{-1}v) = ((h_V^*(z) \otimes h_W(z))f)(v),$$

thus f is an invariant in the representation $h_V^* \otimes h_W$ of $\mathbf{C}^*$ on $(V^* \otimes W)_{\mathbf{R}}$, and hence f is of type $(0,0)$. (In case one has $f(h_V(z)v) = (z\bar{z})^n h_W(z)f(v)$, f is of type (n,n).) Thus a morphism of Hodge structures is a Hodge class and one finds that:

$$Hom_{Hod}(V, W) \cong B(V^* \otimes W).$$

2.6. **Morphisms and correspondences.** We apply the relation between morphisms of Hodge structures and Hodge classes to the cohomology of algebraic varieties. Let V $(\subset H^k(X, \mathbf{Q}))$ be a sub-Hodge structure, the orthogonal projection $\pi_V : H^k(X, \mathbf{Q}) \to V$ (use the polarization on H^k) and the inclusion $i_V : V \hookrightarrow H^k(X, \mathbf{Q})$ are morphisms of Hodge structures. Let W $(\subset H^l(Y, \mathbf{Q}))$ be a sub-Hodge structure isomorphic to V, let $g : V \stackrel{\cong}{\to} W$. Then

$$f := i_W g \pi_V : H^k(X, \mathbf{Q}) \longrightarrow H^l(Y, \mathbf{Q})$$

is a morphism of Hodge structures and hence $f \in B(H^k(X, \mathbf{Q})^* \otimes H^l(Y, \mathbf{Q}))$.

The cohomology of $X \times Y$ can be computed by the Künneth formula:

$$H^n(X \times Y, \mathbf{Q}) = \oplus_{p+q=n} H^p(X, \mathbf{Q}) \otimes H^q(Y, \mathbf{Q}).$$

Poincaré duality shows that $H^p(X, \mathbf{Q}) \cong H^{2d-p}(X, \mathbf{Q})^*$ where $d = \dim X$. Thus f defines a Hodge class in

$$f \in B(H^k(X, \mathbf{Q})^* \otimes H^l(Y, \mathbf{Q})) = B(H^{2d-k}(X, \mathbf{Q}) \otimes H^l(Y, \mathbf{Q}))) \hookrightarrow B(H^{2d-k+l}(X \times Y, \mathbf{Q})).$$

According to the Hodge conjecture, there should exist a cycle Z on $X \times Y$ with cycle class $[Z] = f$. A cycle on $X \times Y$ is also called a correspondence between X and Y.

Thus the Hodge conjecture implies that any morphism of Hodge structures has a geometric origin in a cycle Z on the product of the varieties. This observation is of importance for the theory of (the category of Hodge) motives in which, roughly speaking, the objects are sub-Hodge structures and the morphisms are (equivalence classes of) correspondences.

3. The Mumford-Tate group.

3.1. The Mumford-Tate group $MT(V)$ of a rational Hodge structure (V, h) is an algebraic subgroup of $GL(V)$. It allows one to find the Hodge cycles $B(V^{\otimes m})$, for any m, using representation theory of Lie groups.

3.2. As the Hodge structures obtained from smooth projective varieties are polarized, we restrict ourselves to that case. A polarization Ψ on a weight k Hodge structure (V, h) may be considered as an element of $B(V^* \otimes V^*)$ and the isomorphism

$$V \stackrel{\cong}{\longrightarrow} V^*, \qquad v \longmapsto [w \mapsto \Psi(v, w)]$$

is a morphism of Hodge structures (it is a strict morphism $V \to V^*(-k)$).

An algebraic group G $(\subset GL(V))$ defined over $\mathbf{Q}$ is defined by polynomial equations in the n^2 matrix entries g_{ij} of g and $\det(g)^{-1}$ with coefficients in $\mathbf{Q}$. For a commutative $\mathbf{Q}$-algebra A

(for example $A = \mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$), the set of solutions to these equations in A^{n^2} is a group (under matrix multiplication) and this group is denoted by $G(A)$.

The algebraic subgroup of $GL(V)$ 'fixing' Ψ is defined to be:

$$G(\Psi) := \{g \in GL(V) : \ \Psi(gv, gw) = \nu(g)\Psi(v, w) \quad \text{for some } \nu(g) \in \mathbf{G}_m\}.$$

Here $\mathbf{G}_m$ is the multiplicative group, so $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^*$ etc. In case the weight is odd we usually write $GSp(\Psi)$ and if the weight is even $GO(\Psi)$ for $G(\Psi)$.

3.3. **Definition.** Let (V, Ψ) be a polarized rational Hodge structure of weight k.

1. G_1 is the algebraic subgroup of the $g \in G(\Psi)$ for which there is a $\omega(g) \in \mathbf{G}_m$ such that $g \cdot t = \omega(g)^m t$ for all $t \in B(V^{\otimes m})$.
2. G_2 is the smallest algebraic subgroup of $GL(V)$ which is defined over $\mathbf{Q}$ and which still satisfies
$$h(\mathbf{C}^*) \subseteq G_2(\mathbf{R}).$$

3.4. **Remark.** The condition is that $g \in G_1$ acts as a scalar multiple of the identity on the $\mathbf{Q}$-subspace $B(V^{\otimes m})$ of $V^{\otimes m}$, the shape of the scalar follows from tensoring Hodge cycles. Since $h(\mathbf{C}^*) \in G(\Psi)$ we have $G_2 \subset G(\Psi)$.

3.5. **Theorem.** For any polarized rational Hodge structure (V, h, Ψ) one has:

$$G_1 = G_2 =: \ MT(V) \qquad \text{and} \qquad MT(V) \subset G(\Psi),$$

the algebraic group $MT(V)$ is called the Mumford-Tate group of (V, h, Ψ). The Hodge cycles in the Hodge structure $(V^{\otimes m}, h^{\otimes m})$ are the $MT(V)$-'invariants':

$$B(V^{\otimes m}) = \{w \in V^{\otimes m} : \ gw = \omega(g)^m w \quad \forall g \in MT(V)\}.$$

The Mumford-Tate group is reductive (i.e. every finite dimensional representation of $MT(V)$ is a direct sum of irreducible representations).

Proof. This is all in [DMOS], but we defined $MT(V)$ as the projection in $GL(V)$ of the group G in I.3 and we use the polarization to identify V and V^*. Proposition I.3.4 shows that $G_1 = G_2$ and Proposition I.3.1 identifies the Hodge cycles as $MT(V)$-invariants. The reductivity is proven with Weyl's unitary trick in Proposition I.3.6, again the polarization is essential. □

3.6. **Corollary.** We have: $End_{Hod}(V) \cong End_{MT}(V)$, with

$$End_{MT}(V) = \{M \in End(V) : \ Mg = gM \quad \forall g \in MT(V)\}.$$

Proof. This follows from Theorem 3.5 since $End(V) = V^* \otimes V \cong V \otimes V$, hence $End_{Hod}(V) \cong B(V^{\otimes 2})$ and $End_{MT}(V)$ corresponds to the space of $MT(V)$-invariants in $V^{\otimes 2}$. □

4. Hodge structures of weight one and two

4.1. Given a polarized Hodge structure (V,h,Q) of weight 2, it is interesting to know if there exists a Hodge structure (W,h_W) of weigth 1 with

$$V \hookrightarrow W\otimes W$$

(we do not even require that W be polarized). Note that if there is a pair of weight 1 Hodge structures W_1, W_2 with $V\subset W_1\otimes W_2$ then $V\subset W\otimes W$ with $W=W_1\oplus W_2$.

In general the existence of W depends very much on the Mumford-Tate group of V but in case $MT(V)=GO(Q)$, the answer is very easy. The proof of the following proposition follows §7, Remarques of [D]. Deligne's observation has been further explored by C. Schoen in [S4].

4.2. **Proposition.** Let (V,h,Q) be a polarized Hodge structure of weight 2 with $\dim V^{2,0}>0$ and $MT(V)=GO(Q)$.

If $V\subset W\otimes W$ for some Hodge structure W of weight 1 we must have $\dim V^{2,0}=1$.

Proof. The tangent space at the identity element of an algebraic group is the Lie algebra of that group. The inclusions $SO(Q)\subset GO(Q)=\mathbf{G}_mO(Q)\subset GL(V)$ give inclusion of Lie algebras $so(Q)\subset go(Q)\subset End(V)$. The sub-space $so(Q)$ is defined by the linear equations $\sum_{ij}(\partial F/\partial g_{ij})(I)M_{ij}=0$ where $F\in\mathbf{Q}[\ldots,g_{ij},\ldots]$ runs over the equations defining $SO(Q)$. Since for any $F\in\mathbf{Q}[\ldots,g_{ij},\ldots]$ we have $F(\ldots,\delta_{ij}+\epsilon m_{ij},\ldots)=F(I)+\epsilon\sum_{ij}(\partial F/\partial g_{ij})(I)M_{ij}$ where $I=(\delta_{ij})$ and $\epsilon^2=0$, $so(Q)$ is the subspace of $M\in End(V)$ such that $Q((I+\epsilon M)x,(I+\epsilon M)y)=Q(x,y)$, that is:

$$so(Q)=\{M\in End(V):\ Q(Mx,y)+Q(x,My)=0\quad\forall x,\,y\in V\}.$$

Since $h(z)\in GO(Q)$ one has $h(z)SO(Q)h^{-1}(z)\subset SO(Q)$, the differential of this automorphism is called $Ad(h(z))$. It defines a natural Hodge structure on $so(Q)$:

$$Ad(h):\mathbf{C}^*\longrightarrow GL(so(Q)_{\mathbf{R}}),\qquad z\longmapsto Ad(h(z))=[M\mapsto h(z)Mh(z)^{-1}].$$

Since $h(t)=t^2I_V$ for $t\in\mathbf{R}^*$, the map $Ad(h(t))$ is the identity, hence $so(Q))$ is a Hodge structure of weight 0.

The spaces $V^{2,0}$ and $V^{0,2}$ are isotropic for Q and Q gives a duality between them, see 1.8. So we can choose a basis $f_1,\ldots,f_m$ of $V^{2,0}$ and $f_{m+1},\ldots,f_{2m}$ of $V^{0,2}$ such that $Q(y,y)=y_1y_{m+1}+y_2y_{m+2}+\ldots+y_my_{2m}$ with $y=\sum y_if_i$. In case $m>1$ the following linear map B lies in $so(Q)_{\mathbf{C}}$:

$$B:V_{\mathbf{C}}\longrightarrow V_{\mathbf{C}},\qquad B(V^{1,1})=0,\quad B(f_1)=f_{2m},\quad B(f_m)=-f_{m+1},\quad B(f_i)=0$$

if $i\neq 1,\,m$. Since $h(z)$ acts as $z^p\bar{z}^q$ on $V^{p,q}$ it is easy to verify that

$$h(z)Bh(z)^{-1}=z^{-2}\bar{z}^2B,\qquad\text{hence}\quad so(Q)^{-2,2}\neq 0.$$

Let W be a Hodge structure of weight 1 and let $h_W:\mathbf{C}^*\to GL(W_{\mathbf{R}})$ be the homomorphism which defines the Hodge structure on W. The Hodge structure on $W\otimes W$ is defined by $h_2(z)(u\otimes w)=(h_W(z)u)\otimes(h_W(z)w)$. Thus $MT(W\otimes W)\subset GL(W)$ where $GL(W)$ acts on $W\otimes W$ by $A\cdot(u\otimes w)=(Au)\otimes(Aw)$. Therefore the Lie algebra $Lie(GL(W))=End(W)$ acts on $W\otimes W$ by

$$X\cdot(u\otimes w)=(Xu)\otimes w\,+\,u\otimes(Xw)\qquad(X\in End(W),\ u,\ w\in W).$$

For $X \in End(W)$ the map $Ad(h_2(z))(X)$ $(\in End(W \otimes W))$ is given by:

$$(h_2(z) \cdot X \cdot h_2(z)^{-1}) \cdot (u \otimes w) = (h_W(z) X h_W(z)^{-1} u) \otimes w + u \otimes (h_W(z) X h_W(z)^{-1} w),$$

hence $Ad(h_2(z))(X) = Ad(h_W(z))(X)$ $(= h_W(z) X h_W(z)^{-1} \in End(W))$ acting on $W \otimes W$. The eigenvalues of $h_W(z)$ on $W_{\mathbf{C}} = W^{1,0} \oplus W^{0,1}$ are z and $\bar{z}$, so the eigenvalues of the map $End(W)_{\mathbf{C}} \to End(W)_{\mathbf{C}}$, $X \mapsto h_W(z) X h_W(z)^{-1}$ are $z\bar{z}^{-1}$, 1, $z^{-1}\bar{z}$. Thus the Hodge structure on $End(W)$ $(\subset End(W \otimes W))$ has trivial $(-2,2)$-part. Therefore $so(Q)$ cannot be contained in $End(W)$. $\square$

4.3. We consider the polarized weight two Hodge structures (V, h, Q) with $\dim V^{2,0} = 1$ in some more detail. The quadratic form defined by the polarization on $V_{\mathbf{R}}$ is negative definite on V_2, a two dimensional subspace, and positive definite on V_1 (cf. 1.8).

4.4. **Lemma.** Let (V, Q) be a $\mathbf{Q}$-vector space of dimension n with a bilinear form Q of signature $(2-, (n-2)+)$. Then there is a natural bijection between the following two sets

1. The set of algebraic homomorphisms $h : \mathbf{C}^* \to GO(Q)(\mathbf{R})$ such that (V, h, Q) is a polarized Hodge structure of weight two.
2. The set of oriented two-dimensional subspaces $W \subset V_{\mathbf{R}}$ such that the restriction of Q to W is negative definite.

Proof. Given h, one defines $W := V_2$ and the orientation on V_2 is defined by the basis v, $h(e^{\pi i/4})v$, for (any) $v \in V_2$, $v \neq 0$.

Conversely, given the oriented W, define $h(e^{i\phi})$ to be rotation with angle 2ϕ (in the positive sense) on W and to be the identity on $W^\perp$. For $t \in \mathbf{R}$, let $h(t)$ be scalar multiplication by t^2 on V, this defines the representation $h : \mathbf{C}^* \to GL(V_{\mathbf{R}})$. $\square$

4.5. **Existence of polarized weight two Hodge structures.** The Lemma implies that given a $\mathbf{Q}$-vector space V of dimension n and a non-degenerate quadratic form Q on V with signature $(2-, (n-2)+)$ there exist polarized weight two Hodge structures (V, h, Q) since we can certainly find a $W \subset V_{\mathbf{R}}$ on which Q is negative definite.

There is a basis of $V_{\mathbf{R}}$ such that $Q = -X_1^2 - X_2^2 + X_3^2 + \ldots + X_n^2$. If the restriction of Q to a subspace W is negative definite then we can find a basis a, b of W such that $a = (1, 0, a_3, \ldots a_n) = (1, 0, a')$, $b = (0, 1, b_3, \ldots, b_n) = (0, 1, b')$ (else W contains a non-zero element $(0, 0, c_3, \ldots, c_n)$ contradicting that Q is negative definite on W). Thus W depends on $2(n-2)$-real parameters $(a', b') \in U \subset \mathbf{R}^{2(n-2)}$ for a certain open subset U.

4.6. **Remark.** The group $GO(Q)(\mathbf{R})$ acts in a natural way on both of the sets mentioned in Lemma 4.4:

$$h \longmapsto h^g := [z \mapsto g h(z) g^{-1}], \qquad W \longmapsto gW,$$

this action is compatible with the bijection we indicated. By Witt's theorem, for any two subspaces W, $W' \in V_{\mathbf{R}}$ on which Q is negative definite, there exists a $g \in SO(Q)(\mathbf{R})$ with $gW = W'$ ([La], I.4.2). Moreover, if $\dim V > 2$ we can also find a $g \in SO(Q)(\mathbf{R})$ with $gW = W$ and which reverses the orientation on W. Hence $SO(Q)$ acts transitively on the set of Hodge structures and this set is thus identified with $SO(Q)(\mathbf{R})/S(O(2) \times O(n-2))(\mathbf{R})$ where $n = \dim V > 2$. This set is actually the disjoint union (corresponding to the choice of orientation) of two copies of a Hermitian symmetric domain, in particular, the W depend on $n-2$ complex parameters.

4.7. **Lemma.** Let (V,h,Q) be a weight two Hodge structure with $\dim V^{2,0}=1$. Then, for general h, we have $MT(V)=GO(Q)$.

Proof. The proof is similar to the proof of [vG], 6.11. □

5. From weight two to weight one.

5.1. We recall the construction of Kuga and Satake ([KS], [S1] and [D]) which associates to a polarized Hodge structure (V,h,Q) of weight two, with $\dim V^{2,0}=1$, a polarized Hodge structure $(C^+(Q),h_s,E)$ of weight one. In Proposition 6.3 we show that one can recover the Hodge structure on V from the one on $C^+(Q)$ and that $V\subset C^+(Q)\otimes C^+(Q)$ (inclusion of Hodge structures).

5.2. **Quadratic Forms.** Let (V,h,Q) be a polarized weight two rational Hodge structure with $\dim V^{2,0}=1$, let $n=\dim V$. We will simply write $Q(v)$ for $Q(v,v)$, thus the polarization Q is also viewed as a (non-degenerate) quadratic form Q on the $\mathbf{Q}$-vector space V. Since Q has signature $(2-,(n-2)+)$ there is a basis of V such that ([La], Cor. 2.4):

$$Q:\quad d_1X_1^2+d_2X_2^2+\dots d_nX_n^2,\qquad d_1,\ d_2<0,\quad d_3,\dots,d_n>0.$$

5.3. **Clifford algebras.** Associated to (V,Q) there is a 2^n-dimensional associative $\mathbf{Q}$-algebra, the Clifford algebra $C(Q)$ (cf. [La], Ch. V, [S2], Ch. 9) and a linear injective map $i:V\hookrightarrow C(Q)$. This is characterized by the following universal property:

Let A be a $\mathbf{Q}$-algebra and let $f:V\to A$ be a linear map such that $f(v)^2=Q(v)$ for all $v\in V$. Then there is a unique $\mathbf{Q}$-algebra homomorphism $g:C(Q)\to A$ such that $f=g\circ i$.

We observe that if V has basis $e_1,\dots,e_n$ and Q is given by $\sum d_iX_i^2$ with respect to this basis, one has $e_i^2=Q(e_i)=d_i$ and, for $i\neq j$, $(e_i+e_j)^2=Q(e_i+e_j)=d_i+d_j$ whence:

$$e_i^2=d_i,\qquad e_ie_j=-e_je_i\qquad\text{if}\quad i\neq j.$$

A $\mathbf{Q}$-basis for $C(Q)$ is given by the products

$$e^a:=e_1^{a_1}e_2^{a_2}\dots e_n^{a_n},\qquad a=(a_1,\dots,a_n)\in\{0,1\}^n$$

and

$$C(Q):=\oplus_a\mathbf{Q}e^a.$$

In particular, the vector space V is a subspace of $C(V)$:

$$i:V\hookrightarrow C(V),\qquad e_i\longmapsto e_1^0\dots e_i^1\dots e_n^0,$$

note that V is not a subalgebra however. The even Clifford algebra is the sub-algebra:

$$C^+(Q):=\oplus_a\mathbf{Q}e^a,\qquad a\in\{0,1\}^n,\qquad \sum a_i\equiv\ 0 \bmod 2.$$

Similarly, the vector space $C^-(Q)$ is the span of the e^a with $\sum a_i\equiv 1 \bmod 2$. We will study the Clifford algebra in more detail in section 7.

5.4. **The complex structure on $C^+(Q)_{\mathbf{R}}$.** Let

$$h : \mathbf{C}^* \longrightarrow GO(Q)(\mathbf{R}) \subset GL(V_{\mathbf{R}})$$

be the homomorphism defining the Hodge structure on V. Recall that

$$V_{\mathbf{R}} = V_2 \oplus V_1, \qquad \text{with} \quad V_1 \otimes \mathbf{C} = V^{1,1}, \quad V_2 \otimes \mathbf{C} = V^{2,0} \oplus V^{0,2},$$

and the direct sum is orthogonal for Q. The space V_2 is a real two dimensional vector space.

5.5. **Lemma.** Let $\{f_1, f_2\}$ be a basis of V_2 such that $V^{2,0} = \langle f_1 + if_2 \rangle$ and $Q(f_1) = -1$.

1. For all $x, y \in \mathbf{R}$: $Q(xf_1 + yf_2) = -(x^2 + y^2)$.
2. The element $J := f_1f_2 = -f_2f_1 \in C^+(Q)_{\mathbf{R}}$ satisfies $J^2 = -1$.
3. The element J does not depend on the choice of f_1, f_2 as above.

Proof. Let $\langle v + iw \rangle$ be any basis of $V^{2,0}$ with $v, w \in V_2$. The polarization restricted to V_2 is negative definite, hence we can find a $\lambda \in \mathbf{R}$ with $Q(\lambda v) = \lambda^2 Q(v) = -1$, take $f_1 = \lambda v$, $f_2 = \lambda w$. From Lemma 1.8 we have that $Q = 0$ on $V^{2,0}$, hence $0 = Q(f_1 + if_2, f_1 + if_2) = Q(f_1) - Q(f_2) + 2iQ(f_1, f_2)$, hence also $Q(f_2) = -1$ and f_1, f_2 are perpendicular. It follows that $Q(xf_1 + yf_2) = -(x^2 + y^2)$. Thus in $C(Q)_{\mathbf{R}}$ we have: $(xf_1 + yf_2)^2 = -(x^2 + y^2)$ so $f_1f_2 + f_2f_1 = 0$.

In $C(Q)_{\mathbf{R}}$ we now have $f_i^2 = -1$ and $f_1f_2 = -f_2f_1$, therefore

$$J^2 = (f_1f_2)(f_1f_2) = -f_1^2f_2^2 = -1.$$

Let f_1', f_2' be another such basis of V_2, then there is an orthogonal 2×2 matrix A with $f_i' = Af_i$. Since both $f_1 + if_2$ and $f_1' + if_2'$ span $V^{2,0}$, an eigenspace for all $h(z)$, A must commute with all $h(z)$'s, hence A is a rotation. Therefore $f_1' = af_1 + bf_2$ and $f_2' = -bf_1 + af_2$ with $a^2 + b^2 = 1$. Thus $f_1'f_2' = -ab(-1) + (a^2 + b^2)f_1f_2 + ab(-1) = f_1f_2$. □

5.6. **The weight one Hodge structure on $C^+(Q)$.** With J as in Lemma 5.5 we define a homomorphism

$$h_s : \mathbf{C}^* \longrightarrow GL(C^+(Q)_{\mathbf{R}}), \qquad a + bi \longmapsto a - bJ := [x \longmapsto (a - bJ)x],$$

(with $a, b \in \mathbf{R}$, $x \in C^+(Q)_{\mathbf{R}}$; the '$-$' sign is needed for 6.3). So we let $a - bJ \in C^+(Q)_{\mathbf{R}}$ act by right multiplication on $C^+(Q)_{\mathbf{R}}$. This is obviously an algebraic homomorphism and defines a rational Hodge structure of weight one on the $\mathbf{Q}$-vector space $C^+(Q)$.

5.7. **Polarizations.** We show that the Hodge structure $(C^+(Q), h_s)$ has a polarization. Given $c \in C^+(Q)$ the right multiplication by c is a $\mathbf{Q}$-linear map $C^+(Q) \to C^+(Q)$, $x \mapsto cx$. We denote by $Tr(c)$ $(\in \mathbf{Q})$ the trace of this linear map then

$$Tr : C^+(Q) \longrightarrow \mathbf{Q}$$

is a $\mathbf{Q}$-linear map. There is a $\mathbf{Q}$-linear algebra anti-involution ι on $C^+(V)$ (so $\iota(xy) = \iota(y)\iota(x)$) which is given by ([La], V, 1.11; [S2], 9.3):

$$\iota : C^+(Q) \longrightarrow C^+(Q), \qquad e_1^{a_1} \dots e_n^{a_n} \longmapsto e_n^{a_n} \dots e_1^{a_1} \qquad (a_i \in \{0, 1\}).$$

5.8. **Lemma.** Let $\{e^a\}$ with $a \in \{0,1\}^n$, $\sum a_i \equiv 0 \bmod 2$, be the standard basis of $C^+(Q)$.

1. We have:
$$Tr(e^a) = \begin{cases} 0 & \text{if} \quad a \neq 0, \\ 2^{n-1} & \text{if} \quad a = 0. \end{cases}$$
In particular:
$$Tr(xy) = Tr(yx) \qquad \text{and} \qquad Tr(\iota(x)) = Tr(x).$$
2. For a, $b \in \{0,1\}^n$ we have:
$$Tr(\iota(e^a)e^b) = \begin{cases} 0 & \text{if} \quad a \neq b, \\ 2^{n-1} d_1^{a_1} d_2^{a_2} \dots d_n^{a_n} & \text{if} \quad a = b. \end{cases}$$

Proof. Consider the matrix of multiplication by e^a w.r.t. the standard basis. Since
$$e^a e^b = \lambda e^c, \qquad \text{with} \quad c_i \equiv a_i + b_i \bmod 2$$
and λ is, up to sign, the product of the d_i for which $a_i = b_i = 1$, we see that $e^a e^b$ is a scalar multiple of the basis vector e^c but $e^c = e^b$ only if $a = 0$. Thus $Tr(e^a) = 0$ unless $a = 0$ and then $e^0 = 1$ hence $Tr(e^0) = \dim C^+(Q)$. Since $Tr(e^a e^b) = 0 = Tr(e^b e^a)$ if $a \neq b$ and, obviously, $Tr(e^a e^a) = Tr(e^a e^a)$, we get $Tr(xy) = Tr(yx)$ as $(x, y) \mapsto Tr(xy)$ is bilinear. Finally $\iota(e^a) = \pm e^a$, hence $Tr(\iota(e^a)) = 0 = Tr(e^a)$ if $a \neq 0$ and if $a = 0$ we have $e^a = 1 = \iota(e^a)$.

For the second part, since $\iota(e^a) = \pm e^a$ we have $\iota(e^a)e^b = \mu e^c$ for some $\mu \in \mathbf{Q}$ and $c_i = a_i + b_i \bmod 2$. Thus $Tr(\iota(e^a)e^b) = 0$ unless $a = b$. In that case $\iota(e^a)e^a = d_1^{a_1} \dots d_n^{a_n}$ since:
$$\iota(e_1^{a_1} \dots e_n^{a_n}) e_1^{a_1} \dots e_n^{a_n} = e_n^{a_n} \dots e_1^{a_1} e_1^{a_1} \dots e_n^{a_n} = e_n^{a_n} \dots e_2^{a_2} d_1^{a_1} e_2^{a_2} \dots e_n^{a_n} = \dots = d_1^{a_1} \dots d_n^{a_n}.$$
This concludes the proof of the lemma. □

5.9. **Proposition.** Let $\alpha := \pm e_1 e_2 \in C^+(\mathbf{Q})$. Then the bilinear form:
$$E : C^+(\mathbf{Q}) \times C^+(\mathbf{Q}) \longrightarrow \mathbf{Q}, \qquad E(v, w) := Tr(\alpha \iota(v) w)$$
is a polarization of the weight one Hodge structure $(C^+(\mathbf{Q}), h_s)$ (for suitable choice of sign).

Proof. The form E is obviously $\mathbf{Q}$-bilinear. We have $J = f_1 f_2 = -f_2 f_1 \in C^+(Q)_{\mathbf{R}}$ hence $\iota(J) = -J$. Thus, with $z = a + bi \in \mathbf{C}^*$ we get:
$$\begin{aligned} E(h_s(z)x, h_s(z)y) &= Tr(\alpha\iota((a - bJ)x)(a - bJ)y) \\ &= Tr(\alpha\iota(x)(a + bJ)(a - bJ)y) \\ &= Tr(\alpha\iota(x)(a^2 + b^2)y) \\ &= z\bar{z}E(x, y). \end{aligned}$$
The symmetry of the bilinear form $E(x, h_s(i)y)$ follows from:
$$\begin{aligned} E(x, h_s(i)y) &= -Tr(\alpha\iota(x)Jy) \\ &= -Tr(\iota(\alpha\iota(x)Jy)) \\ &= -Tr(\iota(y)\iota(J)\iota^2(x)\iota(\alpha)) \\ &= -Tr(\iota(y)(-J)x(-\alpha)) \\ &= Tr(\alpha\iota(y)h_s(i)x) \\ &= E(y, h_s(i)x), \end{aligned}$$
here we used $\iota(\alpha) = \iota(e_1 e_2) = e_2 e_1 = -e_1 e_2 = -\alpha$ and Lemma 5.8.

It remains to check that $E(x, h(i)x)$ is either positive or negative definite. First we show that, given (V, Q), it suffices to consider just one Hodge structure (V, h, Q). Let (V, h, Q) and (V, h', Q) be polarized Hodge structures and let J, $J' \in C^+(Q)_{\mathbf{R}}$ be the associated complex structures on $C^+(Q)_{\mathbf{R}}$. From 4.6 we have a $g \in SO(Q)(\mathbf{R})$ with $h' = ghg^{-1}$ and $gV_2 = V_2'$, in particular the $f_1, f_2 \in V_2$ with $J = -f_1f_2 \in C^+(Q)_{\mathbf{R}}$ are mapped to $f_i' = gf_i$ and $J' = -f_1'f_2'$. There is a $\tilde{g} \in C^+(Q)_{\mathbf{R}}$ with $\tilde{g}v\tilde{g}^{-1} = gv$ for all $v \in V$ $(\subset C(Q))$ ([C], Theorem 10.3.1) and $\tilde{g}\iota(\tilde{g}) \in \mathbf{R}^*$ ([C], Cor. 3.1.1, [S2], Lemma 9.3.2). Therefore $\tilde{g}^{-1} = \lambda\tilde{g}$ for a $\lambda \in \mathbf{R}^*$ and $J' = \tilde{g}J\tilde{g}^{-1} = \lambda\tilde{g}J\iota(\tilde{g})$ so:

$$\begin{aligned} E(x, J'x) &= Tr(\alpha\iota(x)J'x) \\ &= \lambda Tr(\alpha\iota(x)\tilde{g}J\iota(\tilde{g})x) \\ &= \lambda Tr(\alpha\iota(y)Jy) \qquad \text{with} \qquad y := \iota(\tilde{g})x \\ &= \lambda E(y, Jy) \end{aligned}$$

hence $E(x, J'x)$ is definite iff $E(x, Jx)$ is definite.

We consider the Hodge structure with $V_{\mathbf{R}} = V_2 \oplus V_1$, $V_2 = \langle e_1, e_2\rangle_{\mathbf{R}}$ (so Q restricited to V_2 is $d_1X_1^2 + d_2X_2^2$ with $d_1, d_2 < 0$) and we take $J = ce_1e_2$ with $c = (d_1d_2)^{-1/2}$ $(\in \mathbf{R}_{>0})$. Then we have:

$$E(x, h(i)y) = -Tr(\alpha\iota(x)Jy) = -cTr(e_1e_2\iota(x)e_1e_2y).$$

We write the basis elements as:

$$e^a = e_1^{a_1}e_2^{a_2}f_a \qquad \text{with} \quad f_a = e_3^{a_3}\dots e_n^{a_n}.$$

Then $\iota(e^a) = \iota(f_a)e_2^{a_2}e_1^{a_1}$. In case $a_1 = a_2 = 0$ we have:

$$\iota(e^a)e_1e_2 = \iota(f_a)e_1e_2 = (-1)^r e_1\iota(f_a)e_2 = (-1)^{2r}\iota(f_a) = e_1e_2\iota(e^a),$$

here $r = a_3 + \ldots + a_n$. Similarly, if $a_1 = 1$, $a_2 = 0$:

$$\iota(e^a)e_1e_2 = \iota(f_a)e_1e_1e_2 = (-1)^r e_1\iota(f_a)e_1e_2 = -(-1)^r e_1\iota(f_a)e_2e_1 = -e_1e_2\iota(e^a),$$

and proceeding in this way one verifies that: $\iota(e^a)e_1e_2 = e_1e_2(-1)^{a_1+a_2}\iota(e^a)$. Therefore:

$$\begin{aligned} E(e^a, h_s(i)e^b) &= -cTr(e_1e_2\iota(e^a)e_1e_2e^b) \\ &= -cTr((e_1e_2)^2(-1)^{a_1+a_2}\iota(e^a)e^b) \\ &= +cd_1d_2(-1)^{a_1+a_2}Tr(\iota(e^a)e^b). \end{aligned}$$

Using the previous lemma we get:

$$E(e^a, h(i)e^b) = \begin{cases} 0 & \text{if} \quad a \neq b, \\ 2^{n-1}(cd_1d_2)((-1)^{a_1+a_2}d_1^{a_1}d_2^{a_2})d_3^{a_3}\dots d_n^{a_n} & \text{if} \quad a = b. \end{cases}$$

Thus $E(e^a, h(i)e^a) > 0$ for all a since $d_1, d_2 < 0$, $d_3, \ldots, d_n > 0$. This, combined with the $\mathbf{Q}$-bilinearity of Tr, proves that $E(x, h(i)x) > 0$ for all $x \in C^+(Q) - \{0\}$. □

6. The Mumford-Tate group of the Kuga-Satake Hodge structure.

6.1. In the previous section we constructed a polarized rational weight one Hodge structure $(C^+(Q), h_s, E)$. We recall some basic facts on this Hodge structure. A detailed study of the Clifford algebra in the next section will give more precise information on the simple sub-Hodge structures of $C^+(Q)$.

6.2. **The spin representation.** Let Q a non-degenerate quadratic form on a F-vector space V (we usually consider the case $F = \mathbf{Q}$ but the cases $F = \mathbf{R}$, $\mathbf{C}$ are also of interest to us). The Spin group $CSpin(Q)$, an algebraic group defined over F, can be defined as the subgroup of $C^+(Q)^*$, the units of the ring $C^+(Q)$:

$$CSpin(Q) = \{g \in C^+(Q)^* : \ gVg^{-1} \subset V\}.$$

By its very definition, we have a homomorphism

$$\rho : CSpin(Q) \longrightarrow GL(V), \qquad g \longmapsto [v \mapsto gvg^{-1}].$$

Since $Q(\rho(g)v) = (\rho(g)v)^2 = gv^2g^{-1} = gQ(v)g^{-1} = Q(v)$, the image of ρ lies in $O(Q)$, the orthogonal group of Q, one actually has: $\rho(CSpin(Q)) = SO(Q)$, ([C], Theorem 10.3.1).

The group $CSpin(Q)$ also acts by multiplication on the left on $C^+(Q)$, this gives a homomorphism, called the spin representation (cf. [S2], 9.3):

$$\sigma : CSpin(Q) \longrightarrow GL(C^+(Q)), \qquad g \longmapsto [x \longmapsto gx].$$

We will identify $CSpin(Q)$ with its image in $GL(C^+(Q))$.

6.3. **Proposition.**

1. The image of the homomorphism $h_s : \mathbf{C}^* \to GL(C^+(Q)_\mathbf{R})$ is contained in the algebraic group $CSpin(Q)(\mathbf{R})$. Hence $MT(C^+(Q)) \subseteq CSpin(Q)$, and
$$MT(C^+(Q)) = CSpin(Q) \qquad \text{if} \quad MT(V) = GO(Q).$$
2. For t, $\phi \in \mathbf{R}$ we have:
$$h(te^{i\phi}) = t^2\rho(h_s(e^{i\phi})),$$
so we recover (V, Q) from the Hodge structure on $C^+(Q)$ defined by h_s.
3. The Hodge structure (V, h) is a sub-Hodge structure of $(C^+(Q) \otimes C^+(Q), h_s \otimes h_s)$:
$$V \hookrightarrow C^+(Q) \otimes C^+(Q).$$

Proof. First we show $h_s(\mathbf{C}^*) \in CSpin(Q)(\mathbf{R})$. Since $h_s(te^{i\phi}) = t^2(a - bJ)$ with $a^2 + b^2 = 1$ and then $(a - bJ)^{-1} = a + bJ$ it suffices to show $(a - bJ)V(a + bJ) \subset V$. For $v \in V_\mathbf{R} \subset C(Q)_\mathbf{R}$ and a, $b \in \mathbf{R}$ we have:

$$(a - bJ)v(a + bJ) = (a^2v - b^2JvJ) + ab(-Jv + vJ).$$

Recall that $V_\mathbf{R} = V_1 \oplus V_2$ (orthogonal sum) and $J = f_1f_2$ with $f_i \in V_2$. Thus $Jv = vJ$ for $v \in V_1$, and hence $(a - bJ)v(a + bJ) = (a^2 + b^2)v \in V$. Note that

$$Jf_1 = (f_1f_2)f_1 = -f_1^2f_2 = f_2 \in V, \qquad Jf_2 = (f_1f_2)f_2 = -f_1 \in V.$$

Similarly, $f_1J = -f_2$, $f_2J = f_1$. This gives:

$$(a - bJ)f_1(a + bJ) = (a^2 - b^2)f_1 - 2abf_2, \qquad (a - bJ)f_2(a + bJ) = (a^2 - b^2)f_2 + 2abf_1$$

hence also $(a - bJ)V_2(a + bJ) \subset V$ and we conclude that $h_s(\mathbf{C}^*) \in CSpin(Q)(\mathbf{R})$.

Moreover, we see that $\rho(h_s(a + bi))v = v$ for $v \in V_1$ and that $f_1 \pm if_2$ is an eigenvector of $\rho(h_s(a + bi))$ $(\in GL(V_\mathbf{R}))$ with eigenvalue $(a \pm ib)^2$, this verifies 6.3.2.

We already saw that $h_s(\mathbf{C}^*) \subset CSpin(Q)(\mathbf{R})$, hence $MT(C^+(Q)) \subseteq CSpin(Q)$. If we assume $MT(V) = GO(Q)$ and G is a subgroup of $SO(Q)$ defined over $\mathbf{Q}$ with $h(e^{i\phi}) \in G(\mathbf{R})$

for all $\phi \in \mathbf{R}$, then G must be equal to $SO(Q)$. Hence $\rho(MT(C^+(Q))) = SO(Q)$. Since $\ker(\rho) \cong \mathbf{G}_m$ ($\subset MT(C^+(Q))$), we get $MT(C^+(Q)) = CSpin(Q)$.

For $g \in CSpin(Q)$ one has $\nu(g) := \iota(g)g \in \mathbf{G}_m$ ([S2], Lemma 9.3.2), hence:

$$E(gv, gw) = Tr(\alpha\iota(v)\iota(g)gw) = \nu(g)E(v, w).$$

Therefore the isomorphism $V \to V^*$ defined by E is equivariant for the action of $CSpin(Q)$ (up to the homomorphism ν). We will identify the $CSpin(Q)$-representations $End(C^+(Q)) = C^+(Q)^* \otimes C^+(Q) \cong C^+(Q) \otimes C^+(Q)$.

We choose an invertible element, say e_1, in V ($\subset C(Q)$). Then we have an inclusion:

$$V \hookrightarrow End(C^+(Q)), \qquad v \longmapsto M_v := [y \mapsto vye_1].$$

The image of V in $End(C^+(Q))$ is a sub-representation on which $CSpin(Q)$ acts via ρ:

$$(gM_vg^{-1})(y) = (gM_v)(g^{-1}y) = g(vg^{-1}ye_1) = (gvg^{-1})ye_1 = M_{gvg^{-1}}(y) = M_{\rho(g)v}y.$$

As $h_s(\mathbf{C}^*) \subset CSpin(Q)(\mathbf{R})$ it follows that $V \hookrightarrow C^+(Q) \otimes C^+(Q)$ is sub-Hodge structure. $\square$

6.4. The spin representation is not irreducible in general. In fact, for any $x, y \in C^+(Q)$ we have $\sigma(g)(xy) = gxy = (\sigma(g)x)y$, so the $\mathbf{Q}$-linear maps $C^+(Q) \to C^+(Q)$, $x \mapsto xy$ commute with $CSpin(Q)$. Therefore we have an injective map

$$C^+(Q) \hookrightarrow End_{CSpin}(C^+(Q)) \quad (:= \{M \in End(C^+(Q)) :\ M\sigma(g) = \sigma(g)M\ \}).$$

(Due to the action on the right, we should write $C^+(Q)^{op}$, but $C^+(Q)^{op} \cong C^+(Q)$, cf. [La], V, Prop. 1.11.) These are all the maps which commute with the $CSpin(Q)$ representation:

6.5. **Lemma.** We have:

$$C^+(Q) \cong End_{CSpin(Q)}(C^+(Q)).$$

Proof. The group $CSpin(Q)$ is an algebraic subgroup of $GL(C^+(Q))$ and thus its Lie algebra, $cspin(Q)$ is a subalgebra of $End(C^+(Q))$ and $cspin(Q) \otimes_{\mathbf{Q}} \mathbf{C}$ is the Lie algebra of the complex Lie group $CSpin(Q)(\mathbf{C})$. Moreover, $End_{CSpin(Q)}(C^+(Q)) \cong End_{cspin(Q)}(C^+(Q))$. The latter is the subspace of $C^+(Q)$ defined by:

$$End_{cspin(Q)}(C^+(Q)) = \{X \in End(C^+(Q)) :\ XM - MX = 0 \quad \forall M \in cspin(Q)\}.$$

Considering these equations for $X \in End(C^+(Q)_{\mathbf{C}})$ we see that

$$End_{cspin(Q)}(C^+(Q)) \otimes_{\mathbf{Q}} \mathbf{C} \cong End_{cspin(Q)\otimes_{\mathbf{Q}}\mathbf{C}}(C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}).$$

From the representation theory of complex Lie algebras (see 6.6 below for the case n even) we know that

$$End_{cspin(Q)\otimes_{\mathbf{Q}}\mathbf{C}}(C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}) \cong C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C},$$

we use that $C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}$ is the even Clifford algebra of the quadratic form Q on $V \otimes_{\mathbf{Q}} \mathbf{C}$. Thus $\dim_{\mathbf{Q}} End_{cspin(Q)}(C^+(Q)) = \dim_{\mathbf{C}} C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C} = 2^{n-1}$, hence $C^+(Q)$ must be all of $End_{cspin(Q)}(C^+(Q))$ for dimension reasons. $\square$

6.6. **Example.** In case Q is the quadratic form $Y_1Y_{m+1} + \ldots + Y_mY_{2m}$ (this is the case over a finite extension of $\mathbf{Q}$), the Clifford algebra and the spin representation can be described as follows (cf. [FH], p. 304; they consider the Clifford algebra over $\mathbf{C}$, but the same arguments work over any extension of $\mathbf{Q}$).

Let Z be the m-dimensional subspace of V defined by $Y_{m+1} = \ldots = Y_{2m} = 0$ Then

$$C(Q) \cong End(\Lambda^* Z), \qquad \Lambda^* Z = \mathbf{Q} \oplus Z \oplus \wedge^2 Z \oplus \ldots$$

The even Clifford algebra is identified with the subalgebra

$$C^+(Q) \cong End(\Lambda^{even} Z) \times End(\Lambda^{odd} Z) \cong M_{2^{m-1}}(\mathbf{C}) \times M_{2^{m-1}}(\mathbf{C}).$$

Since $C^+(Q) \subset End_{cspin(Q)}(C^+(Q))$, the spin representation must, at least, split in the direct sum of 2^{m-1} copies of a representation σ_+ and 2^{m-1} copies of a representation σ_-. By considering the Lie algebra action one can see that σ_+ and σ_- are both irreducible and are not isomorphic (see [FH]). This implies that $C^+(Q) = End_{cspin(Q)}(C^+(Q))$. The representations σ_+ and σ_-, each of dimension 2^{m-1}, are the half spin representations of $so(Q)$.

7. Clifford algebras

7.1. We recall that the Clifford algebra can be built up from quaternion algebras. If $Q = d_1X_1^2 + d_2X_2^2 + \ldots d_nX_n^2$, with $d_i \in \mathbf{Q}$, we write:

$$V = \langle d_1 \rangle \oplus \ldots \oplus \langle d_n \rangle.$$

7.2. **Quaternion Algebras.** The quaternion algebra $A = (a,b)_F$ over a field F has an F-basis $1,\ i,\ j,\ k$ such that

$$A = F \oplus Fi \oplus Fj \oplus Fk, \qquad i^2 = a, \quad j^2 = b, \quad ij = k = -ji, \qquad a, b \in F^*,$$

with $F^* = F - \{0\}$ and $xa = ax$ for all $x \in F$, $a \in A$. In case $F = \mathbf{Q}$ we omit the index F. Note however that $(a,b)_F$ and $(c,d)_F$ may be isomorphic without $a = c$ and $b = d$. For example, $(a,b)_F \cong (b,a)_F$ (via $i \mapsto j,\ j \mapsto i,\ k \mapsto -k$). The algebra $M_2(F)$ of 2×2 matrices over F is a quaternion algebra and for any $b \in F^*$

$$M_2(F) \cong (1,b)_F, \qquad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

The quaternion algebra $(a,b)_F$ is either a (skew) field or is isomorphic to to the matrix algebra $M_2(F)$. One has: $(a,b)_F \cong M_2(F)$ if and only if the equation $ax^2 + by^2 = abz^2$ has a non-trivial solution $(x,y,z) \in F^3$ ([La], III,2.7; [S2], Corollary 2.11.10; note that if $ax^2 + by^2 - abz^2 = 0$ then $(xi + yj + zk)^2 = 0$ hence $(a,b)_F$ is not a field).

7.3. **Example.** In case $Q = dX^2$, $d \in \mathbf{Q}^*$, the Clifford algebra is isomorphic to $\mathbf{Q}[T]/(T^2 - d)$, which is a field if d is not a square and is isomorphic to $\mathbf{Q} \times \mathbf{Q}$ if d is a square. In this case $C^+(Q) \cong \mathbf{Q} = M_1(\mathbf{Q})$.

7.4. **Example.** In case $Q = aX_1^2 + bX_2^2$, $C(Q)$ is the quaternion algebra (a,b). The even Clifford algebra $C^+(Q) = \mathbf{Q} \oplus \mathbf{Q}e_1e_2$ with $(e_1e_2)(e_1e_2) = -e_1^2e_2^2 = -ab$, so if $-ab$ is not a square, the even Clifford algebra is a field: $C^+(Q) \cong \mathbf{Q}(\sqrt{-ab})$. In case $-ab = c^2$ is a square, $C^+(Q) \cong \mathbf{Q} \times \mathbf{Q}$, $e_1e_2 \mapsto (c, -c)$.

7.5. **Example.** Let $Q = d_1X_1^2 + d_2X_2^2 + d_3X_3^2$, note that $1,\ i := e_1e_2,\ j := e_2e_3,\ k := d_2e_1e_3$, satisfy the rules of the quaternion algebra $(-d_1d_2, -d_2d_3)$ and that $1, \dots, k$ are a basis of $C^+(Q)$ hence

$$C^+(Q) \cong (-d_1d_2, -d_2d_3).$$

The element $z = e_1e_2e_3$ is in the center of the Clifford algebra $C(Q)$ (since it commutes with the generators e_1, e_2, e_3) and $z^2 = -e_1^2e_2^2e_3^2 = -d_1d_2d_3$. Moreover, $C(Q) = C^+(Q) \oplus zC^+(Q)$, hence $C(Q) \cong C^+(Q) \otimes \mathbf{Q}(\sqrt{-d_1d_2d_3})$. Since $\mathbf{Q}(\sqrt{-d_1d_2d_3}) \cong C(-d_1d_2d_3X^2)$, we have ([S2], Lemma 9.2.9, with some changes in notation):

$$C(d_1X_1^2 + d_2X_2^2 + d_3X_3^2) \cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3\rangle).$$

7.6. **Graded tensor products.** To determine the Clifford algebra of the quadratic form $Q = d_1X_1^2 + \dots + d_nX_n^2$ on V, so $V = \langle d_1\rangle \oplus \dots \oplus \langle d_n\rangle$, we use that if $V = V' \oplus V''$ is the orthogonal direct sum with $Q_{|V'} = Q'$ and $Q_{|V''} = Q''$ then ([S2], 9.2.5):

$$C(Q' \oplus Q'') = C(Q')\hat{\otimes}C(Q'').$$

Here $\hat{\otimes}$, the graded tensor product, is the tensor product of the underlying $\mathbf{Q}$-vector spaces but the product is given by ([S2], 9.1.4)) $(x\hat{\otimes}y)(x'\hat{\otimes}y') = \epsilon xx'\hat{\otimes}yy'$ with $x, x' \in C^\pm(Q')$ and $y, y' \in C^\pm(Q'')$ and $\epsilon = 1$ except if $y \in C^-(Q'')$ and $x' \in C^-(Q')$, in that case $\epsilon = -1$.

An easy example is the case $Q = aX^2 + bY^2$, $Q' = \langle a\rangle$ and $Q'' = \langle b\rangle$, in that case $C(Q' \oplus Q'') = C(Q')\hat{\otimes}C(Q'') \cong (a, b)$ as in example 7.4. Using example 7.5 as induction step we then obtain, from Theorem 9.2.10 of [S2] and a basic fact on quaternion algebras, the following result:

7.7. **Theorem.** Let $Q = \sum d_iX_i^2$, with $d_i \in \mathbf{Q}^*$. Then we have:

1. In case $n = 2m$, let $d := (-1)^m d_1 \dots d_n$. Then the even Clifford algebra $C^+(Q)$ is isomorphic to one of the following two algebras
   (a) In case $\sqrt{d} \in \mathbf{Q}$ we have $C^+(Q) \cong M_{2^{m-2}}(D) \times M_{2^{m-2}}(D)$ with a quaternion algebra D over $\mathbf{Q}$.
   (b) In case $\sqrt{d} \notin \mathbf{Q}$ we have $C^+(Q) \cong M_{2^{m-2}}(D) = M_{2^{m-2}}(\mathbf{Q}) \otimes_{\mathbf{Q}} D$, where D is a quaternion algebra over a quadratic field extension $F = \mathbf{Q}(\sqrt{d})$ of $\mathbf{Q}$.
2. In case $n = 2m+1$, we have $C^+(Q) \cong M_{2^{m-1}}(D)$ for a quaternion algebra D over $\mathbf{Q}$.

Note that if $D \cong M_2(\mathbf{Q})$ then $M_{2^a}(D) \cong M_{2^{a+1}}(\mathbf{Q})$.

Proof. For $n \leq 3$ see 7.3, 7.4, 7.5. For $n > 3$ use Example 7.5:

$$\begin{aligned} C(\langle d_1\rangle \oplus \dots \oplus \langle d_n\rangle) &\cong C(\langle d_1\rangle \oplus \dots \oplus \langle d_3\rangle)\hat{\otimes}C(\langle d_4\rangle \oplus \dots \oplus \langle d_n\rangle) \\ &\cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3\rangle)\hat{\otimes}C(\langle d_4\rangle \oplus \dots \oplus \langle d_n\rangle) \\ &\cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3\rangle \oplus \langle d_4\rangle \oplus \dots \oplus \langle d_n\rangle). \end{aligned}$$

in the last lines the first tensor product is the usual one since all elements in $(-d_1d_2, -d_2d_3)$ are even elements of $C(\langle d_1\rangle \oplus \langle d_2\rangle \oplus \langle d_3\rangle)$.

In case $n = 4$ one thus finds $C(Q) \cong (-d_1d_2, -d_2d_3) \otimes (-d_1d_2d_3, d_4)$ and $C^+(Q) \cong (-d_1d_2, -d_2d_3) \otimes F$, with $F \cong \mathbf{Q}[X]/(X^2 - d_1d_2d_3d_4)$ the even Clifford algebra of $\langle -d_1d_2d_3\rangle \oplus \langle d_4\rangle$. So F is either a field or is isomorphic to $\mathbf{Q} \times \mathbf{Q}$.

We continue this game and we find quaternion algebras $A_{13}, A_{15}, \dots A_{1,2k+1}$, such that

$$C(Q) \cong A_{13} \otimes A_{15} \otimes \dots A_{1,2k+1} \otimes C(\langle (-1)^k d_1d_2 \dots d_{2k+1}\rangle \oplus \langle d_{2k+2}\rangle \oplus \dots \langle d_n\rangle),$$

and the even Clifford algebra is:

$$C^+(Q) \cong A_{13} \otimes A_{15} \otimes \ldots A_{1,2k+1} \otimes C^+(\langle (-1)^k d_1 d_2 \ldots d_{2k+1}\rangle \oplus \langle d_{2k+2}\rangle \oplus \ldots \langle d_n\rangle).$$

We conclude that if $n = 2m$, then $C(Q)$ is the tensor product of $m-1$ quaternion algebras which is then graded tensored with an m-th quaternion algebra. Thus $C^+(Q)$ is a tensor product of $m-1$ quaternion algebras with an algebra $\mathbf{Q}[X]/(X^2-d)$ and it is not hard to see that $d = (-1)^m d_1 \ldots d_{2m}$.

In case $n = 2m+1$, $C(Q)$ is the tensor product of $2m$ quaternion algebras (which is $C^+(Q)$) graded tensored by $C(\langle (-1)^m d_1 \ldots d_{2m+1}\rangle)$.

Now we recall that the tensor product $A \otimes_{\mathbf{Q}} B$ of two quaternion algebras A, B over $\mathbf{Q}$ is isomorphic with $M_2(D) = M_2(\mathbf{Q}) \otimes_{\mathbf{Q}} D$ for some quaternion algebra D. Let $A = (a,c)$, $B = (b,d)$ then, in A, we have

$$(x_1 i + x_2 j + x_3 k)^2 = a x_1^2 + c x_2^2 + ac x_3^2.$$

The quadratic form in 6 variables $ax_1^2 + cx_2^2 + acx_3^2 - (by_1^2 + dy_2^2 + bdy_3^2)$ is indefinite for any choice of signs for $a, \ldots, d$. Hence Meyer's theorem ([S5], Corollary 4.3.2) implies that it has a non-trivial zero. Thus there are $x \in A$, $y \in B$ with $x^2 = y^2 = e$, so if A and B are fields they have the quadratic field $K = \mathbf{Q}(\sqrt{e})$ in common. Then A and B are cyclic algebra's over K: $A = (K/\mathbf{Q}, r)$, $B = (K/\mathbf{Q}, s)$ (so $A = K \oplus K\alpha$ with $\alpha^2 = r$ and $\alpha x = \bar{x}\alpha$ where '$^-$' is the conjugation on K). In [La], III, 2.11 and [S2], Theorem 8.12.7 one finds an explicit isomorphism $A \otimes_{\mathbf{Q}} B \cong M_2(D)$ with $D = (K/\mathbf{Q}, rs)$. $\square$

8. Kuga-Satake varieties

8.1. The rational, polarized, weight one Hodge structure $(C^+(Q), h_s, E)$ defines an isogeny class of abelian varieties. Each variety in this isogeny class is called a Kuga-Satake variety for (V, h, Q). More precisely, consider the dual W of the 2^{n-2}-dimensional complex vector space $C^+(Q)^{1,0}$:

$$W := \left(C^+(Q)^{1,0}\right)^*, \qquad \text{let} \quad \Gamma \subset C^+(Q)^*$$

be a free $\mathbf{Z}$-module with $\Gamma \otimes_{\mathbf{Z}} \mathbf{Q} = C^+(Q)^*$, the dual of the $\mathbf{Q}$-vector space $C^+(Q)$. The image of Γ under the projection from $(C^+(Q)^{0,1})^*$: $\Gamma \subset C^+(Q)^*_{\mathbf{C}} \to W$ is a lattice in the complex vector space W and the quotient is an abelian variety A_Γ (with Riemann form E_Γ defined by E), which is a Kuga-Satake variety of (V, h, Q). There is a natural isomorphism of rational, polarized, weight one Hodge structures:

$$(H^1(A_\Gamma, \mathbf{Q}), E_\Gamma) \cong (C^+(Q), h_s, E).$$

For an abelian variety A one has $End(A) \otimes \mathbf{Q} \cong End_{Hod}(H^1(A, \mathbf{Q}))$ and if

$$End(A) \otimes_{\mathbf{Z}} \mathbf{Q} \cong M_{n_1}(D_1) \times \ldots \times M_{n_d}(D_d) \qquad \text{then} \quad A \sim A_1^{n_1} \times \ldots \times A_d^{n_d}$$

where the D_i are (skew) fields, $\sim$ means isogeneous and the A_i are simple abelian varieites. Thus Theorem 7.7 gives the decomposition in simple factors of a Kuga-Satake variety in case $MT(V) = GO(Q)$.

8.2. **Example.** Let (V, h, Q) be a three dimensional rational Hodge structure of weight two with $\dim V^{2,0} = 1$ and $MT(V) = GO(Q)$. Assume that with respect to some basis of V we can write $Q = -X_1^2 - X_2^2 + dX_3^2$. Then from Example 7.5 we know:

$$C^+(Q) \cong (-1, d).$$

A Kuga-Satake variety of (V, h, Q) is an abelian surface X with $End(X) \cong C^+(Q)$. Hence it is a product of two isogeneous elliptic curves if $C^+(Q)$ is not a field, otherwise it is a simple abelian surface. Note that $C^+(Q) \cong M_2(\mathbf{Q})$ iff $-x^2 + dy^2 = -dz^2$ has a non-trivial solution in $\mathbf{Q}^3$. This happens for example if $d = 1$. In case $d \equiv 3 \mod 4$ we get a field, this is best seen by multiplying the equation $-x^2 + dy^2 = -dz^2$ by d and substituting $y := d^{-1}y$, $z := d^{-1}z$ to get $dx^2 = y^2 + z^2$. If this had a non-trivial solution, we had one in $\mathbf{Z}^3$ (multiply by product of denominators) and after dividing by a power of 2 at least one of x, y, z would be odd. Now consider the equation mod 4 and use that a square is either 0 or 1 mod 4.

8.3. **Geometric realizations.** In 4.5 we observed that given (V, Q), one can find weight two Hodge structures (V, h, Q). Since $V \subset C^+(Q) \otimes C^+(Q)$ we find an inclusion of Hodge structures:

$$V \hookrightarrow H^1(A_\Gamma, \mathbf{Q}) \otimes H^1(A_\Gamma, \mathbf{Q}) \subset H^2(A_\Gamma \times A_\Gamma, \mathbf{Q}).$$

Thus any polarized weight two Hodge structure (V, h, Q) with $\dim V^{2,0} = 1$ is a sub-Hodge structure of the cohomology of some algebraic variety.

This is *not* true if $\dim V^{2,0} > 1$, Griffiths work on variations of Hodge structures ('Griffiths transversality') implies that the general polarized Hodge structure of weight two (and similar results hold for higher weight) with $\dim H^{2,0} > 1$ is not a sub-Hodge structure of the cohomology of an algebraic variety, see [CKT] and [Ma].

9. Abelian varieties of Weil type and Kuga-Satake varieties

9.1. The Hodge conjecture for abelian varieties of dimension at least 4 is still open (see [G] for a recent overview). For 4 dimensional abelian varieties (abelian 4-folds) Moonen and Zarhin [MZ1], [MZ2] proved that if the Hodge conjecture is true for abelian 4-folds of Weil type, then the Hodge conjecture is true for all abelian 4-folds.

An abelian $2n$-fold A is of Weil type if $End(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ contains an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-d})$, $d > 0$ and the action of the endomorphism $\sqrt{-d}$ on the tangent space at the origin of A has eigenvalues $\sqrt{-d}$, $-\sqrt{-d}$, each with multiplicity n. Associated to the pair (A, K) is a 'discriminant' $\delta \in \mathbf{Q}^*/N(K^*)$ where $N(K^*)$ is the group of the norms of K^*, that is, the elements $a^2 + b^2 d$, a, $b \in \mathbf{Q}$ (see [vG] for an introduction to these varieties).

The Hodge conjecture for the general abelian $2n$-fold of Weil type with $d = 3$, $\delta = 1$ is proved by C. Schoen in [S3] for $n = 2$. The case $n = 3$ is done in [S5]. There he also proves that the Hodge conjecture for the general abelian 4-fold of Weil type with $d = 3$ (and any δ) follows from this by specializing the 6-fold to a product of a 4-fold and an abelian surface.

The following result, due to G. Lombardo, may also be of some interest. The case that $K = \mathbf{Q}(\sqrt{-1})$ was considered by Paranjape [P] and, from a different point of view, by Matsumoto and others ([M] and references given there).

9.2. **Theorem.** Let $(A, K = \mathbf{Q}(\sqrt{-d}))$ be an abelian 4-fold of Weil type with discriminant $\delta = 1$. Then A^4 is a Kuga-Satake variety of a weight two Hodge structure (V, h, Q) with

$$Q = -X_1^2 - X_2^2 + X_3^2 + X_4^2 + X_5^2 + dX_6^2.$$

Conversely, if (V, h, Q) is a weight two polarized Hodge structure with Q as above, then the Kuga-Satake variety of (V, h, Q) is isogeneous to A^4 with A an abelian 4-fold of Weil type.

Proof. See [Lo], we only verify that the Kuga-Satake variety of a general (V, h, Q) is isogeneous to A^4 with A an abelian 4-fold with $K = End(A)$. Using the rules from the proof of Theorem 7.7, one finds:

$$C(Q) \cong (-1, 1) \otimes C(-X^2 + X_4^2 + X_5^2 + dX_6^2) \cong M_2(\mathbf{Q}) \otimes C(-X^2 + X_4^2 + X_5^2 + dX_6^2),$$

since $(-1, 1) \cong (1, -1) \cong M_2(\mathbf{Q})$, cf. 7.2. Next:

$$C(-X^2 + X_4^2 + X_5^2 + dX_6^2) \cong (1, 1) \otimes \Big(C(\langle 1\rangle)\hat{\otimes}C(\langle d\rangle)\Big) \cong M_2(\mathbf{Q}) \otimes C(X^2 + dY^2),$$

since $C^+(X^2 + dY^2) \cong \mathbf{Q}(\sqrt{-d}) \cong K$ we get:

$$C^+(Q) \cong M_2(\mathbf{Q}) \otimes M_2(\mathbf{Q}) \otimes C^+(X^2 + dY^2) \cong M_4(\mathbf{Q}) \otimes K.$$

Hence for general (V, h, Q) any Kuga-Satake variety is isogeneous to 4 copies of a simple abelian variety A, so $\dim A = 4$, with $End(A) \cong K$. $\square$

10. The Kuga-Satake-Hodge conjecture

10.1. Given a polarized Hodge structure (V, h, Q) of weight two with $\dim V^{2,0} = 1$ there exists an abelian variety A, the Kuga-Satake variety of (V, h, Q), with the property that V is a sub-Hodge structure of $H^2(A \times A, \mathbf{Q})$, cf. section 8.3. In case there is another algebraic variety X with $V \hookrightarrow H^2(X, \mathbf{Q})$, the Hodge conjecture predicts (cf. section 2.6) the existence of an algebraic cycle $Z \subset A^2 \times X$, the Kuga-Satake-Deligne correspondence, which realizes the morphism of Hodge structures

$$f : H^2(A \times A, \mathbf{Q}) \xrightarrow{\pi} V \xrightarrow{i} H^2(X, \mathbf{Q}).$$

(More generally, one can consider $V \hookrightarrow H^{2+2n}(X, \mathbf{Q})(n)$.)

A particular case where this happens is when the algebraic variety X has $\dim H^{2,0}(X, \mathbf{Q}) = 1$. The space of the Hodge cycles $B(H^2(X, \mathbf{Q}))$ a sub-Hodge structure of $H^2(X, \mathbf{Q})$, it is the image of $CH^1(X)_{\mathbf{Q}}$ and is called the Neron-Severi group of X (tensored by $\mathbf{Q}$). Let V be the orthogonal complement in $H^2(X, \mathbf{Q})$:

$$H^2(X, \mathbf{Q}) = V \oplus NS(X)_{\mathbf{Q}}.$$

Then V, with the polarization Q induced by the one on $H^2(X, \mathbf{Q})$, is of the type we consider. It has the additional convenient property that it is simple in the sense that if $W \subset V$ is a sub-Hodge structure, then $W = 0$ or $W = V$. In fact, $V = W \oplus W^\perp$ and if $W^{2,0} = 0$ then $W \subset NS(X)_{\mathbf{Q}} \cap V = 0$, else $(W^\perp)^{2,0} = 0$ so $W^\perp = 0$ and $W = V$.

Since for an ample divisor Y on a variety X the restriction map $H^2(X, \mathbf{Q}) \hookrightarrow H^2(Y, \mathbf{Q})$ is injective if $\dim Y \geq 2$ (Lefschetz Theorem), we may assume without loss of generality that X is a surface. Then the Hodge conjecture leads to:

10.2. **Kuga-Satake-Hodge conjecture.** Let X be a smooth surface with $\dim H^{2,0}(X) = 1$ and let $H^2(X, \mathbf{Q}) = V \oplus NS(X)_{\mathbf{Q}}$. Let Q be the induced polarization on V and let A be a Kuga-Satake variety of (V, h, Q).

Then there exist a surface Z and a diagram

$$\begin{array}{ccc} Z & \stackrel{\phi}{\longrightarrow} & A \times A \\ \pi \downarrow & & \\ X & & \end{array} \qquad \text{such that} \quad \pi_* \phi^* : H^2(A \times A, \mathbf{Q}) \longrightarrow H^2(X, \mathbf{Q})$$

induces an isomorphism $V \stackrel{\cong}{\longrightarrow} V$.

10.3. We show that conjecture 10.2 is indeed a special case of the Hodge conjecture. Let $f : H^2(A^2, \mathbf{Q}) \to H^2(X, \mathbf{Q})$ be the morphism of Hodge structures as in 10.1. Then $f \in B(H^{2d}(A^2 \times X, \mathbf{Q}))$ with $d = \dim A^2$ (cf. 2.6). The Hodge conjecture implies that f should be the class of a codimension d cycle $\sum_i a_i Z_i$ on the $d+2$-dimensional variety $A^2 \times X$, hence each Z_i is a surface (an irreducible 2 dimensional variety). Each Z_i defines a morphism of Hodge structures $[Z_i] : H^2(A^2, \mathbf{Q}) \to H^2(X, \mathbf{Q})$, since V is simple the restriction of each $[Z_i]$ to V is either 0 or is an isomorphism on its image. As f is an isomorphism, there is at least one Z_i such that $[Z_i]$ induces an isomorphism, take Z to be that Z_i.

10.4. Assume that Z exists. It is easy to see that we may replace Z by its desingularization. The map $\pi : Z \to X$ must be surjective (because $H^{2,0}(X) \subset \pi_* H^2(Z, \mathbf{C})$). Since $H^2(A^2, \mathbf{Q}) = \Lambda^2 H^1(A^2, \mathbf{Q})$ and pull-back is compatible with cup product, we get:

$$V \hookrightarrow \pi_* \mathrm{Image}\left(\Lambda^2 H^1(Z, \mathbf{Q}) \longrightarrow H^2(Z, \mathbf{Q})\right).$$

Conversely, given a surface Z with a surjective map $\pi : Z \to X$ having the above property, the albanese map $Z \to Alb(Z)$ is essentially a map $\phi : Z \to A^2$ with $\pi_* \phi^*$ as in 10.2.

In the example of Paranjape [P] and in Example 11.3 below the surface Z is a product of two curves, also in the example of Voisin [V] it seems to be possible to choose for Z a product of curves. In all these examples the surfaces X are K3 surfaces (so $\dim H^2(X, \mathbf{Q}) = 22$, $\dim H^{2,0}(X) = 1$), but their Neron-Severi groups are rather large.

11. Hodge structures and imaginary quadratic fields.

11.1. The example of C. Voisin [V] deals with a polarized weight two Hodge structure (V, h, Q) with $\dim V^{2,0} = 1$ which has an automorphism ϕ of order 3 preserving the polarization: $Q(v, w) = Q(\phi v, \phi w)$ (it is induced from an automorphism of a $K3$ surface). The action of ϕ on V gives V the structure of a vector space over $K = \mathbf{Q}(\sqrt{-3})$.

Voisin asked for the simple factors of the Kuga-Satake variety of (V, h, Q). She already proved that the (isogeny class of an) elliptic curve A_0 with complex multiplication by K and an abelian variety A_1 with $2 \dim A_1 = \dim V$, $K \subset End(A_1) \otimes \mathbf{Q}$, are simple components. The following theorem (an exercise in representation theory) gives the simple factors in general.

11.2. **Theorem.** Let (V, h, Q) be a polarized weight two Hodge structure with $\dim V^{2,0} = 1$. Let $K \subset End_{Hod}(V)$ be an imaginary quadratic field such that V is K-vector space and assume $Q(xv, xw) = x\bar{x} Q(v, w)$ for $x \in K$, v, $w \in V$. Let $n = 2m = \dim V$.

Then there is a Hodge structure S such that

$$C^+(Q) \cong S^{2^{m-2}}, \qquad \dim_{\mathbf{Q}} S = 2^{m+1}.$$

The Hodge structure S splits as:

$$S \cong S_0 \oplus S_1 \oplus \ldots \oplus S_m, \qquad S_i \cong S_{m-i}, \qquad \dim_{\mathbf{Q}} S_i = 2\binom{m}{i}.$$

The S_i are simple Hodge structures except S_l if $2l = m$, $l \equiv 2\ (4)$ and the polarization satisfies an additional condition, in that case $S_l \cong (S_l')^2$ and S_l' is irreducible.

The S_i are K-vector spaces, $K \subset End_{Hod}(S_i)$, they are simple if h is general and $V \hookrightarrow S_0 \otimes S_1$ $(\subset C^+(Q) \otimes C^+(Q))$.

Proof. To appear. □

11.3. **Example.** (With thanks to M. Nori.) Smooth quartic surfaces in $\mathbf{P}^3$ are K3 surfaces. Consider a surface X in $\mathbf{P}^3$ defined by:

$$X: \qquad T^4 = F(X, Y, Z)$$

where $F = 0$ defines a smooth plane quartic curve C $(\subset \mathbf{P}^2)$. Let $p \in C$ and let L_p $(\subset \mathbf{P}^2)$ be the tangent line to C at L_p. Then $L_p \cap C = \{p, p', p''\}$ (since p has multiplicty 2). If $p \in C$ is general and u is a suitable parameter along L_p, then restriction of F to L_p is given by $u^2(u-1)(u+1)$ (with $u = 0$ corresponding to $p \in L_p$). The curve in X lying over L_p is defined by $T^4 = u^2(u^2-1)$, thus it is irreducible and its normalization E_p has a $4:1$ map onto $L_p \cong \mathbf{P}^1$ which is totally ramified over $u = \pm 1$ and over $u = 0$ it has two ramification points. Hence E_p is an elliptic curve and since it has an isomorphism of order 4, $T \mapsto iT$, $i^2 = -1$, with fixed points (the points over $u = \pm 1$) one has $E_p \cong E_i := \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$. The elliptic surface

$$\mathcal{E} \longrightarrow C, \qquad \mathcal{E} := \cup_{p \in C} E_p \times \{p\} \ \subset \mathbf{P}^3 \times C,$$

is thus isotrivial and projection on the first factor gives a surjective map $\mathcal{E} \to X$. After a suitable base change $C' \to C$ and normalization the surface will be a product $E_i \times C'$. Thus we have a surjective rational map:

$$\pi: E_i \times C' \longrightarrow X,$$

so V $(\subset H^2(X, \mathbf{Q}))$ lies in the image of π_*. Now $S_0 \cong H^1(E_i, \mathbf{Q})$, hence E_i is isogeneous to a simple factor A_0 of the Kuga-Satake variety A of $V \subset H^2(X, \mathbf{Q})$ and one verifies that $S_1 \subset H^1(C', \mathbf{Q})$, hence $Jac(C')$ maps onto a simple factor A_1 of A. Then we may take $Z = E_i \times C'$ (or a blow up if π is not a morphism), the map ϕ is the composition:

$$\phi: E_i \times C' \longrightarrow E' \times Jac(C') \longrightarrow A_0 \times A_1 \hookrightarrow A \times A.$$

11.4. The following result was inspired by [V] (and was generalized following a discussion with M. Nori). It has interesting geometrical applications and gives a better understanding of the Hodge structures S_i.

11.5. **Theorem.** Let (V,h,Ψ) be a polarized Hodge structure of weight k, let $K \subset End_{Hod}(V)$ be an imaginary quadratic field such that V is a K-vector space and assume $\Psi(xv,xw) = x\bar{x}\Psi(v,w)$ for $x \in K$, v, $w \in V$.

If K acts via scalar multiplication on $V^{k,0}$, then there exist (V,h',Ψ'), (K,h_K,E_K), polarized Hodge structures of weight $k-1$ and one respectively, such that

$$(V,h) \hookrightarrow (V,h') \otimes (K,h_K), \qquad \text{and} \qquad \Psi = (\Psi' \otimes E_K)_{|V}.$$

Proof. A generalization, with K a CM-field, is to appear. □

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