COMPLEX VARIETIES - A.Y. 2011-2012
HOMEWORK

1.1. (a) Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X and let $\mathcal{F}^+$ be the sheaf generated by this presheaf.

Show that the natural homomorphism of sheaves $\tau : \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism $\tau_a : \mathcal{F}_a \to \mathcal{F}^+_a$ on the stalks for all $a \in X$.
(b) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Prove that there is an isomorphism between the sheaves $\mathrm{Im}(\varphi)$ and $\mathcal{F}/\ker(\varphi)$.

1.2. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on a topological space X. Let $X = \cup U_i$ be an open covering of X. Notice that if $V \subset U_i$ is an open subset (for the induced topology on U_i), then V is also open in X. Thus we get sheaves $\mathcal{F}_i$ on U_i by $\mathcal{F}_i(V) := \mathcal{F}(V)$. Let $\phi_i : \mathcal{F}_i \to \mathcal{G}_i$ be homomorphisms of sheaves such that $\phi_i = \phi_j$ on $U_i \cap U_j$ (i.e. $\phi_{i,V} = \phi_{j,V}$ for $V \subset U_i \cap U_j$).

Define a homomorphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that for $V \subset U_i$ one has $\phi_V = \phi_{i,V} : \mathcal{F}(V) \to \mathcal{G}(V)$.

1.3. Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X.

(a) For an open subset $U \subseteq X$ and a section $s \in \mathcal{F}(U)$ define

$$\mathrm{Supp}(s) = \{a \in U : \; s_a \neq 0\},$$

where s_a is the germ of s in the stalk $\mathcal{F}_a$. Prove that $\mathrm{Supp}(s)$ is a closed subset of U.

(b) Let $Z \subseteq X$ be a closed subset. Define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\mathcal{F}(X)$ consisting of all sections whose support is contained in Z. Show that the presheaf

$$V \mapsto \Gamma_{V \cap Z}(V, \mathcal{F}|_V)$$

is a sheaf.

1.4. Consider the sheaves $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$.

(a) Show that any global section $s \in \mathcal{O}_{\mathbb{P}^1}(1)(\mathbb{P}^1)$ yields an injective morphism of sheaves

$$\varphi : \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$$

(b) Describe the stalks of the corresponding quotient sheaf $\mathcal{O}_{\mathbb{P}^1}(1)/\mathcal{O}_{\mathbb{P}^1}$.

1.5. Let $X \subset \mathbb{P}^2$ be a one-dimensional complex manifold defined by a homogeneous polynomial of degree $d > 3$.
(a) Show that $\omega_X = \Omega^1_X$ has a global section ω which is not identically zero.
(b) Show that any holomorphic map $\phi : \mathbb{P}^1 \to X$ must be constant.

1.6. (a) Let $A = (a_{ij})$ be an invertible $(n+1) \times (n+1)$ matrix with complex coefficients. Show that the map

$$\alpha : \mathbb{P}^n \longrightarrow \mathbb{P}^n, \qquad (x_0 : \ldots : x_n) \longmapsto (y_0 : \ldots : y_n), \quad y_i := \sum_{j=1}^{n} a_{ij} x_j$$

(so α is the map induced by $A : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}$) is a biholomomorphic map.
(b) Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Show that

$$\beta : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad (x : y : z) \longmapsto (u : v : w) := (\lambda^2 x : \lambda^3 y : z),$$

is a biholomorphic map and that the elliptic curves E, E' with (affine) equations

$$y^2 = 4x^3 - g_2 x - g_3, \qquad v^2 = 4u^3 - \lambda^4 g_2 u - \lambda^6 g_3,$$

respectively, are isomorphic.
(c) Show that the curves in $\mathbb{P}^2$ defined by

$$x^3 + y^3 + z^3 = 0 \qquad y^2 = 4x^3 - g_3$$

are isomorphic, for any $g_3 \in \mathbb{C}$, $g_3 \neq 0$, and that these curves are also isomorphic to the complex torus $\mathbb{C}/\Lambda$ where $\Lambda = \{n + m\omega : n, m \in \mathbf{Z},\ \omega^3 = 1,\ \omega \neq 1\}$. (Hint: substitute $x = u + v$, $y = u - v$ in the Fermat equation and use affine coordinates with $u = 1$).

1.7. Let E be the elliptic curve in $\mathbb{P}^2$ defined by the (affine) equation

$$y^2 = 4x^3 - g_3, \qquad (g_3 \neq 0).$$

Let $\mathcal{O} := (0 : 1 : 0)$ be the neutral element in the group law on E.
(a) Show that the points $P_\pm$ with affine coordinates $(x, y) = (0, \pm\sqrt{-g_3})$ are points of order three.
(b) Let g_3 be choosen in such a way that the map $F : \mathbb{C} \to \mathbb{P}^2$, $z \mapsto (\wp(z) : \wp'(z) : 1)$, where $\wp$ is the Weierstrass $\wp$-function for the lattice $\Lambda = \{n + m\omega : n, m \in \mathbf{Z},\ \omega^3 = 1,\ \omega \neq 1\}$ as in Exercise 1.6 has image E. Show that

$$(\wp(\omega z) : \wp'(\omega z) : 1) = (\omega\wp(z) : \wp'(z) : 1)$$

for all $z \in \mathbb{C}$. Conclude that the image of $(1 - \omega)/3 \in \mathbb{C}$ under F is P_+ or P_-.

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