COMPLEX MANIFOLDS, ALGANT HOMEWORK

Consign at least one, and preferably not more than 3, of the exercises at least 24 hours before the exam.

1.1. Let $T := \mathbb{C}/\Lambda$ be a complex torus and let $\pi : \mathbb{C} \to T$ be the quotient map. Let $\wp$ be the Weierstrass $\wp$-function for the lattice Λ and let $g_2, g_3 \in \mathbb{C}$ be such that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.
(a) Show that the following subset $C \subset \mathbb{P}^3$, an intersection of the two quadrics,

$$C = \{(x : y : z : t) \in \mathbb{P}^3 : \ y^2 = 4xt - g_2xz - g_3z^2, \quad x^2 = zt\},$$

is a submanifold of $\mathbb{P}^3$.
(b) Show that the map

$$\psi : T \longrightarrow \mathbb{P}^3, \qquad t = \pi(z) \longmapsto (\wp(z) : \wp'(z) : 1 : \wp^2(z))$$

for $t \neq 0$ and $\psi(0) = (0 : 0 : 0 : 1)$ is a holomorphic map and that $\psi(T)$ is isomorphic to T.
(c) Show that $C = \psi(T)$, hence that

$$C \cong T .$$

1.2. (a) Let $A = (a_{ij})$ be an invertible $(n+1) \times (n+1)$ matrix with complex coefficients. Show that the map

$$\alpha : \mathbb{P}^n \longrightarrow \mathbb{P}^n, \qquad (x_0 : \ldots : x_n) \longmapsto (y_0 : \ldots : y_n), \quad y_i := \sum_{j=0}^{n} a_{ij}x_j$$

(so α is the map induced by $A : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}$) is a biholomorphic map.
(b) Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Show that

$$\beta : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad (x : y : z) \longmapsto (u : v : w) := (\lambda^2 x : \lambda^3 y : z),$$

is a biholomorphic map and that the elliptic curves E, E' with (affine) equations

$$y^2 = 4x^3 - g_2x - g_3, \qquad v^2 = 4u^3 - \lambda^4 g_2 u - \lambda^6 g_3,$$

respectively, are isomorphic.
(c) Show that the curves in $\mathbb{P}^2$ defined by

$$x^3 + y^3 + z^3 = 0 \qquad y^2 = 4x^3 - g_3$$

are isomorphic, for any $g_3 \in \mathbb{C}$, $g_3 \neq 0$, and that these curves are also isomorphic to the complex torus $\mathbb{C}/\Lambda$ where $\Lambda = \{n + m\omega : \ n, m \in \mathbf{Z}, \ \omega^3 = 1, \ \omega \neq 1\}$. (Hint: substitute $x = u + v$, $y = u - v$ in the Fermat equation and use affine coordinates with $u = 1$).

1.3. Let E be the elliptic curve in $\mathbb{P}^2$ defined by the (affine) equation

$$y^2 = 4x^3 - g_3, \qquad (g_3 \neq 0).$$

Let $\mathcal{O} := (0:1:0)$ be the neutral element in the group law on E.

(a) Show that the points $P_\pm$ with affine coordinates $(x, y) = (0, \pm\sqrt{-g_3})$ are points of order three.

(b) Let g_3 be chosen in such a way that the map $F : \mathbb{C} \to \mathbb{P}^2$, $z \mapsto (\wp(z) : \wp'(z) : 1)$, where $\wp$ is the Weierstrass $\wp$-function for the lattice $\Lambda = \{n + m\omega : n, m \in \mathbf{Z},\ \omega^3 = 1,\ \omega \neq 1\}$ as in Exercise 1.2 has image E. Show that

$$(\wp(\omega z) : \wp'(\omega z) : 1) = (\omega\wp(z) : \wp'(z) : 1)$$

for all $z \in \mathbb{C}$. Conclude that the image of $(1 - \omega)/3 \in \mathbb{C}$ under F is P_+ or P_-.

1.4. Let $T = \mathbb{C}/\Lambda$ be a complex torus and let $0 \in T$ be the neutral element. Let (U, z) be a complex chart of T with $0 \in U$ and $z(0) = 0$, let $V := T - \{0\}$ and let $g_{UV} := z : U \cap V \to \mathbb{C} - \{0\}$. Let L be the line bundle on T defined by $\{U_0, V, g_{UV}\}$. The k-fold tensor product of L is denoted by $L^{\otimes k}$. A global section s of $L^{\otimes k}$ defines two holomorphic functions s_U, s_V on U, V respectively such that $s_U = g_{UV} s_V$ on $U \cap V$.

(a) Show that $L \cong L_0$, the line bundle defined by the codimension one submanifold $\{0\}$ of T.

(b) For $k \in \mathbf{Z}$ we define a vector space by

$$V_k := \{f : T \to \mathbb{C} :\ f \text{ is meromorphic on } T, \text{ holomorphic on } V \text{ and } ord_0(f) \geq -k\}.$$

Show that $\Gamma(T, L^{\otimes k}) \cong V_k$.

(c) Show that $\dim \Gamma(T, L^{\otimes k}) = 0$ if $k < 0$ and that it is k for $k > 0$. (Hint: use the elliptic functions $\wp, \wp'$ to show there exists an $f \in V_k$ with a pole of order exactly $k (> 1)$ in $0 \in T$.)

1.5. Let X be a complex manifold and let

$$0 \longrightarrow E \longrightarrow F \longrightarrow F/E \longrightarrow 0$$

be an exact sequence of vector bundles on X. Let L be a line bundle on X.

Show that there is an sequence of vector bundles

$$0 \longrightarrow E \otimes L \longrightarrow F \otimes L \longrightarrow (F/E) \otimes L \longrightarrow 0.$$

1.6. Recall that for $k \in \mathbf{Z}$ we have the line bundle $L(k)$ on $\mathbb{P}^1$. In particular, $L(0) \cong \mathbb{P}^1 \times \mathbb{C}$ is the trivial bundle and $L(-1)$ is the tautological bundle, which, by definition, is a subbundle of $L(0)^2 := L(0) \oplus L(0) \cong \mathbb{P}^1 \times \mathbb{C}^2$.

(a) Show that there is an exact sequence

$$0 \longrightarrow L(-1) \longrightarrow L(0)^2 \longrightarrow L(1) \longrightarrow 0.$$

(b) Show that the vector bundles $L(0)^2$ and $L(-1) \oplus L(1)$ are not isomorphic (Hint: consider their global sections).

1.7. Let X be a complex manifold of dimension n with atlas $\{(U_\alpha, z_\alpha)\}$ and denote by $F_{\alpha\beta} : z_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^n$ be the change of coordinates. Then the canonical bundle ω_X of X has transition functions $g_{\alpha\beta} := \det({}^t(JF_{\alpha\beta})^{-1}) = \det(JF_{\alpha\beta})^{-1}$.

(a) The canonical bundle is a complex manifold of dimension $n+1$, with a surjective map $p : \omega_X \to X$. Show that for a suitable atlas of ω_X, the transition functions of ω_X are

$$G_{\alpha\beta} : \mathbb{C} \times z_\beta(U_\alpha \cap U_\beta) \to \mathbb{C} \times \mathbb{C}^n, \qquad (t, u) \longmapsto (g_{\alpha\beta}(u)t, F_{\alpha\beta}(u)).$$

(b) Show that ω, the canonical bundle of the complex manifold ω_X, is the trivial bundle.

(c) The image of the zero section $s : X \to \omega_X$ of the line bundle ω_X is a smooth, codimension one, submanifold of ω_X which we denote by S. Recall that L_S is the line bundle defined by S on ω_X.
The complex manifold ω_X has the open covering $\omega_X = \cup p^{-1}U_\alpha$. We define a line bundle $p^*\omega_X$ on ω_X by the data $\{p^{-1}U_\alpha, p^*g_{\alpha\beta} := g_{\alpha\beta} \circ p\}$.
Show that the line bundles L_S and $p^*\omega_X$ are isomorphic.

1.8. A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open $U \subset X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$.

(a) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves. Show that

$$\mathcal{F}'(U) \ := \ \ker(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

defines a subsheaf on X.

(b) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Let $\mathcal{G}$ be the presheaf defined by $\mathcal{G}(U) := \mathcal{F}(U)/\mathcal{F}'(U)$, with restriction maps induced by those of $\mathcal{F}$. Show that

$$\mathcal{G}_a \ \cong \ \mathcal{F}_a/\mathcal{F}'_a \qquad \text{for all} \quad a \in X\ .$$

(c) Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X and let $\mathcal{F}^+$ be the sheaf generated by this presheaf.
Show that the natural homomorphism of sheaves $\tau : \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism $\tau_a : \mathcal{F}_a \to \mathcal{F}^+_a$ on the stalks for all $a \in X$.

1.9. Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X.

(a) For an open subset $U \subseteq X$ and a section $s \in \mathcal{F}(U)$ define

$$\mathrm{Supp}(s) \ = \ \{a \in U : \ s_a \neq 0\ \},$$

where s_a is the germ of s in the stalk $\mathcal{F}_a$. Prove that $\mathrm{Supp}(s)$ is a closed subset of U.

(b) Let $Z \subseteq X$ be a closed subset. Define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\mathcal{F}(X)$ consisting of all sections whose support is contained in Z. Show that the presheaf

$$V \mapsto \Gamma_{V\cap Z}(V, \mathcal{F}|_V)$$

is a sheaf.

1.10. Consider the sheaves $\mathcal{O}_{\mathbb{P}^1}(d)$ on $\mathbb{P}^1$, where $\mathcal{O}_{\mathbb{P}^1}(d)(U)$ are the holomorphic functions on $\pi^{-1}(U)$ which are homogeneous of degree d and where $\pi : \mathbb{C}^2 - \{0\} \to \mathbb{P}^1$ is the quotient map. One can show that the sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$ is isomorphic to the sheaf of global sections of the line bundle $L(d)$ on $\mathbb{P}^1$. The homogeneous coordinates on $\mathbb{P}^1$ are $(x_0 : x_1)$ so $x_0, x_1 \in \mathcal{O}_{\mathbb{P}^1}(1)(\mathbb{P}^1)$.

(a) Show that

$$\varphi : \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d+1), \qquad \varphi_U(f) := x_0 f, \qquad (\, f \in \mathcal{O}_{\mathbb{P}^1}(d)(U)\,),$$

is an injective homomorphism of sheaves.

(b) Describe the stalks of the corresponding quotient sheaf $\mathcal{Q} := \mathcal{O}_{\mathbb{P}^1}(d+1)/\mathcal{O}_{\mathbb{P}^1}(d)$ and conclude that $\mathcal{Q}$ is the skyscraper sheaf, with group $\mathbb{C}$, concentrated in the point $p := (0 : 1) \in \mathbb{P}^1$.

(c) Show that a skyscraper sheaf is a soft sheaf.

(d) Using the long exact cohomology sequence associated to the exact sequence of sheaves on $\mathbb{P}^1$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d+1) \longrightarrow \mathcal{Q} \longrightarrow 0\ ,$$

for $d = -1$, show that $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

(e) Show that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$ and next that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \neq 0$ for all $d \leq -2$.

1.11. In this exercise we determine the vector space $\Gamma(\mathbb{P}^n, L(d))$ of global sections of the line bundle $L(d)$ on $\mathbb{P}^n$.

Let Y be the submanifold of $\mathbb{P}^n$ defined by $z_n = 0$, where $(z_0 : \ldots : z_n)$ are the homogeneous coordinates on $\mathbb{P}^n$. Notice that Y is isomorphic to $\mathbb{P}^{n-1}$.

(a) Show that $\{(U_j, f_j := z_n/z_j)\}_{0\leq j\leq n}$ are local equations of Y, with the standard open subsets $U_j := \{z = (z_0 : \ldots : z_n) \in \mathbb{P}^n :\ z_j \neq 0\}$.

(b) Let s be a global (holomorphic) section of the line bundle $L(d)$ on $\mathbb{P}^n$ and let $\{s_j : U_j \to \mathbb{C}\}_{0\leq j\leq n}$ be the local sections defined by s and the standard trivialization of $L(d)$, so each s_j is holomorphic on U_j and $s_j(z) = (z_k/z_j)^d s_k(z)$ on $U_j \cap U_k$ for $0 \leq j, k \leq n$.

Assume that $s(z) = 0$ for all $z \in Y$. Show that $s = z_n t$ for a global (holomorphic) section t of $L(d-1)$. (Hint: show that $s_j = (z_n/z_j)t_j$ for a holomorphic function t_j on U_j).

(c) Show that the (restriction) map i^* is well defined, where

$$i^* : \Gamma(\mathbb{P}^n, L(d)) \to \Gamma(\mathbb{P}^{n-1}, L(d)), \quad s = \{(s_j : U_j \to \mathbb{C}\}_{0\leq j\leq n} \mapsto \hat{s} := \{(s_j : U_j \cap Y \to \mathbb{C})\}_{0\leq j\leq n-1}$$

and that the kernel of the linear map i^* is isomorphic to $\Gamma(\mathbb{P}^n, L(d-1))$.

(d) Let $\mathbb{C}[x_0, \ldots, x_n]_d$ be the complex vector space of homogeneous polynomials of degree d in $n+1$-variables. Show that the linear map

$$j = j_{n,d} : \mathbb{C}[x_0, \ldots, x_n]_d \longrightarrow \Gamma(\mathbb{P}^n, L(d)), \qquad F \longmapsto s_F := \{(F(z_0/z_j, \ldots, z_n/z_j) : U_j \to \mathbb{C})\}_{0\leq j\leq n}$$

is well defined.

(e) Consider the case $n = 1$, so $Y = (1 : 0) \in \mathbb{P}^1$ is a point. Thus the line bundle $L(d)$ on $Y \cong \mathbb{P}^0$ is just $Y \times \mathbb{C} \cong \mathbb{C}$ and $\Gamma(\mathbb{P}^0, L(d)) = \mathbb{C}$ for all d.

Show that $i^* : \Gamma(\mathbb{P}^1, L(d)) \to \Gamma(\mathbb{P}^0, L(d))$ is surjective (hint: consider $i^*(s_F)$ where $F = x_0^d$).

Conclude, with induction on d, that $j_{1,d}$ is an isomorphism for all $d \geq 0$ and show that $\dim \Gamma(\mathbb{P}^1, L(d)) = 0$ for $d < 0$.

(f) Assume that $j_{n-1,d}$ is an isomorphism for all $d \geq 0$ and show that the maps $i^* : \Gamma(\mathbb{P}^n, L(d)) \to \Gamma(\mathbb{P}^{n-1}, L(d))$ are surjective for all $d \geq 0$.

(g) Conclude that $j_{n,d} : \mathbb{C}[x_0, \ldots, x_n]_d \longrightarrow \Gamma(\mathbb{P}^n, L(d))$ is an isomorphism for all $n, d \geq 0$ and that $\Gamma(\mathbb{P}^n, L(d)) = 0$ for $d < 0$.