SOME REMARKS ON BRAUER GROUPS OF K3 SURFACES

BERT VAN GEEMEN

Abstract. We discuss the geometry of the genus one fibrations associated to an elliptic fibration on a K3 surface. We show that the two-torsion subgroup of the Brauer group of a general elliptic fibration is naturally isomorphic to the two torsion of the Jacobian of a curve associated to the fibration. We remark that this is related to Recillas' trigonal construction. Finally we discuss the two-torsion in the Brauer group of a general K3 surface with a polarisation of degree two.

Elements of the Brauer group of a projective variety have various geometric incarnations, like Azumaya algebras and Severi-Brauer varieties. Moreover, an element α in the Brauer group of a projective K3 surface X determines a polarized Hodge substructure $T_{<\alpha>}$ of the transcendental lattice T_X of X. We are interested in finding geometric realizations of the Hodge structure $T_{<\alpha>}$ and in relating such a realization to the other incarnations. In this paper we collect some examples and add a few new results. Recently, there has been much interest in Brauer groups of K3 surfaces ([C2], [DP], [Wi]), but our aim is much more modest.

For certain elements $\alpha \in Br(X)$ the Hodge structure $T_{<\alpha>}$ is the transcendental lattice of another K3 surface Y. The surface Y is not necessarily unique. Two K3 surfaces with isomorphic transcendental lattice are said to be Fourier-Mukai partners. Results from Mukai and Orlov show that if Z is a K3 surface then T_Y and T_Z are Hodge isometric if and only if the bounded derived categories of coherent sheaves on Y and Z are isomorphic. The number of Fourier-Mukai partners of a given K3 surface is determined in [HLOY], see also [St].

In sections 1 and 2 we briefly recall some basic definitions. In section 3 we consider the case when the K3 surface X has an elliptic fibration (with a section). In that case $T_{<\alpha>}$ is always the transcendental lattice of a K3 surface Y, but Y is far from unique in general. These surfaces are studied in section 4. They have a fibration in genus one curves (without a section). In fact, any pair of elements $\pm\alpha \in Br(X)$ determines a unique fibration in genus one curves on such a surface. Somewhat surprisingly, Y often has more than one genus one fibration. We use well-known results on K3 surfaces to determine the number of these surfaces, their automorphisms and the genus one fibrations. In Example 4.9 we work out the numerology for certain cyclic subgroups of $Br(X)$. In section 5 we consider projective models for some genus one fibrations.

In section 6 we prove that the two torsion in the Brauer group of a general elliptic fibration $X \to \mathbf{P}^1$ is naturally isomorphic to the two torsion in the Jacobian of a certain curve in X. We observe that the pairs X, Y, where Y is the surface associated to an element α of order two in $Br(X)$, are closely related to Recillas' trigonal construction. This construction is a bijection between trigonal curves with a point of order two and tetragonal curves. In section 7 we remark that classical formulas allow one to give the conic bundle, and hence the Azumaya algebra, on X associated to α.

In the last section we consider K3 surfaces with Néron-Severi group of rank one and we characterize the elements α of order two in the Brauer group such that $T_{<\alpha>}$ is the transcendental lattice of another K3 surface. In case the K3 surface is a double plane, we also consider the other elements of order two. It is well-known that some of these are related to cubic fourfolds, in this way one obtains a geometric realization of $T_{<\alpha>}$. We briefly discuss the remaining case in 8.8.

1. Brauer groups

1.1. **Basic definitions and properties.** The main references for Brauer groups of algebraic varieties and, more generally, schemes are Grothendieck's papers [Gr] and Milne's book [Mec], useful summaries are given in [C1], [C2], [DP]. For recent work on non-projective complex varieties see [Sc].

The cohomological Brauer group of a scheme X is $Br'(X)_{\text{ét}} = H^2_{\text{ét}}(X, \mathcal{O}_X^*)$. For a smooth complex algebraic variety X one has $Br'(X)_{\text{ét}} \cong H^2(X, \mathcal{O}_X^*)_{tors}$, the subgroup of torsion elements in $H^2(X, \mathcal{O}_X^*)$.

An Azumaya algebra is a sheaf of $\mathcal{O}_X$-algebras over X, which is locally isomorphic to $\mathcal{E}nd(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$. Two Azumaya algebras are (Morita) equivalent if $\mathcal{A} \otimes \mathcal{E}nd(\mathcal{E}) \cong \mathcal{A}' \otimes \mathcal{E}nd(\mathcal{E}')$ where $\mathcal{E}, \mathcal{E}'$ are locally free sheaves on X. The Brauer group $Br(X)$ of a complex algebraic variety X is the group of equivalence classes of Azumaya algebras with tensor product.

The Brauer group also classifies Severi-Brauer varieties, these are projective bundles P over X ([A]).

For smooth curves one has $Br(X) = Br'(X)_{\text{ét}} = 0$. For a smooth surface one has $Br(X) = Br'(X)_{\text{ét}}$ and recently A.J. de Jong [dJ] proved that an element of order n in $Br(X)$ has a representative which is an Azumaya algebra locally isomorphic to $\mathcal{E}nd(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank n.

1.2. **Brauer groups of elliptic fibrations.** Let X be a smooth algebraic surface with an elliptic fibration $p : X \to \mathbf{P}^1$ (with a section). The Brauer group of X is isomorphic to the Tate-Shafarevich group $Ш(X) = H^1(\mathbf{P}^1, X^\sharp)$ where $X^\sharp$ is the sheaf of groups on $\mathbf{P}^1$ of local sections of $p_X : X \to \mathbf{P}^1$ ([CD] Ch. 5, [DG], [C1], Theorem 4.4.1, cf. [FM], 1.5.1):

$$Br(X) = Ш(X).$$

A non-trivial element of order n in $Ш(X)$ is a genus one fibration $Y \to \mathbf{P}^1$ (without a section) and n is the minimal degree of a multisection of $p : Y \to \mathbf{P}^1$. There is a birational action of X on Y. The genus one fibrations corresponding to α and $-\alpha \in Ш(X)$ are isomorphic as fibrations ([C1], Remark 4.5.3), but the action of X on them is different. For the spaces corresponding to $k\alpha$, $k \in \mathbf{Z}$, see [DP], §2.2, [C1], Thm 4.5.2. The fibration $Y/\mathbf{P}^1$ has a relative Jacobian fibration which is $X/\mathbf{P}^1$.

2. The Brauer group of a K3 surface

2.1. For a smooth surface with $H^3(X, \mathbf{Z}) = 0$, the exponential sequence shows that

$$Br(X) = H^2(X, \mathcal{O}_X^*)_{tors} \cong \Big(H^2(X, \mathbf{Z})/c_1(Pic(X))\Big) \otimes (\mathbf{Q}/\mathbf{Z}),$$

where $c_1 : Pic(X) \to H^2(X, \mathbf{Z})$ is the first Chern class. In case X is K3 surface, $Pic(X) \cong c_1(Pic(X)) = NS(X)$, the Néron-Severi group of X. The transcendental lattice of X is the sublattice of $H^2(X, \mathbf{Z})$ perpendicular to $NS(X)$ w.r.t. the intersection pairing:

$$T_X = NS(X)^\perp = \{v \in H^2(X, \mathbf{Z}) : vn = 0 \text{ for all } n \in NS(X)\,\}.$$

The intersection pairing is unimodular, in fact,

$$H^2(X, \mathbf{Z}) \cong \Lambda_{K3} := U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

where U is the hyperbolic plane (cf. [BPV], VIII.1). In particular $H^2(X, \mathbf{Z})$ is self-dual, and as $H^2(X, \mathbf{Z})/T_X$ is torsion free (i.e. T_X is a primitive sublattice) we get

$$H^2(X, \mathbf{Z})/NS(X) \xrightarrow{\cong} T_X^* := Hom(T_X, \mathbf{Z}), \qquad v + NS(X) \longmapsto [t \mapsto vt]$$

for $v \in H^2(X, \mathbf{Z})$, $t \in T_X$. Thus for a K3 surface X we obtain:

$$Br(X) \cong T_X^* \otimes (\mathbf{Q}/\mathbf{Z}) \cong Hom(T_X, \mathbf{Q}/\mathbf{Z}).$$

2.2. **Hodge structures.** Let X be a K3 surface. The Hodge structure on $H^2(X, \mathbf{Z})$ is determined by the one dimensional subspace $H^{2,0}(X) =< \omega_X >\subset H^2(X, \mathbf{Z}) \otimes \mathbf{C}$. A Hodge isometry $f : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ is a $\mathbf{Z}$-linear map which preserves the intersection forms and whose $\mathbf{C}$-linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$.

The (weak) Torelli theorem for projective K3 surfaces ([BPV], VIII.6.3) asserts that if $H^2(X, \mathbf{Z})$ and $H^2(Y, \mathbf{Z})$ are Hodge isometric, where X and Y are K3 surfaces, then X and Y are isomorphic.

The period ω_X of X satisfies $\omega_X \omega_X = 0$ and $\omega_X \bar{\omega}_X > 0$. The 'surjectivity of the period map' ([BPV], VIII.14) asserts that given an $\omega \in \Lambda_{K3} \otimes \mathbf{C}$ with $\omega\omega = 0$ and $\omega\bar{\omega} > 0$ there exists a K3 surface X and a Hodge isometry $f : (\Lambda_{K3}, \omega) \to H^2(X, \mathbf{Z})$ (so $< f_{\mathbf{C}}(\omega) >= H^{2,0}(X)$). The Néron-Severi group of X is isomorphic to the lattice $\{v \in \Lambda_{K3} : v\omega = 0\}$. The K3 surface X is projective iff $NS(X)$ contains a class v with $v^2 > 0$, equivalently, the signature of $T_X = NS(X)^\perp$ is $(2+, n-)$ for some n, $0 \leq n \leq 19$.

The transcendental lattice T_X is a polarized Hodge structure (note that $< \omega_X >\subset T_X \otimes \mathbf{C}$). A sublattice $\Gamma \subset T_X$ of finite index is also a polarized Hodge structure, in fact $< \omega_X >\subset \Gamma \otimes \mathbf{C} = T_X \otimes \mathbf{C}$. In case Γ admits a primitive embedding $i : \Gamma \hookrightarrow \Lambda_{K3}$, the vector $i_{\mathbf{C}}(\omega_X) \subset \Lambda_{K3} \otimes \mathbf{C}$ defines a K3 surface $Y = Y(\Gamma, i)$ with period $\omega_Y = i_{\mathbf{C}}(\omega_X)$ and $T_Y = i(\Gamma)$. If X is projective, Y is also projective (the transcendental lattices have the same signatures).

2.3. **Moduli spaces.** Conversely, let Y be a K3 surface and let $T_Y \hookrightarrow T$ be a Hodge isometry such that T/T_Y is cyclic. Mukai [Mu1] has shown that there is a K3 surface X, actually X is a moduli space of sheaves on Y, such that $T \cong T_X$.

Kondō [Ko1], [Ko2] studied the map defined by such inclusions from the moduli space of K3 surfaces with a polarization of degree dn^2 to the one of degree d, this map has a high degree. We are interested in going in the opposite direction, part of the extra data needed for the inverse is a cyclic subgroup of the Brauer group of X.

2.4. **Brauer groups and Hodge substructures.** An element α of order n in $Br(X)$ defines a surjective homomorphism $\alpha : T_X \to \mathbf{Z}/n\mathbf{Z}$ (cf. 2.1) and thus defines a sublattice of index n of Γ. Conversely, any sublattice Γ' of index n in T_X with cyclic quotient group Γ/Γ' determines a subgroup of order n in $Br(X)$. The sublattice defined by the cyclic subgroup $\langle\alpha\rangle$ of $Br(X)$ generated by α is denoted by

$$T_{<\alpha>} = \ker(\alpha : T_X \longrightarrow \mathbf{Q}/\mathbf{Z}) \qquad (\alpha \in Br(X) = Hom(T_X, \mathbf{Q}/\mathbf{Z})).$$

In case there exists a primitive embedding $i : T_{<\alpha>} \hookrightarrow \Lambda_{K3}$, we obtain a K3 surface $Y = Y(T_{<\alpha>}, i)$ as in 2.2. Such an embedding $i : T_{<\alpha>} \hookrightarrow \Lambda_{K3}$ is not unique up to isometry in general. In fact, even the orthogonal complement of the images under the various embeddings need not be isomorphic as lattices (cf. [O], [HLOY], [St]).

2.5. **Lattices.** We recall some basic definitions ([Dol], [N]). Given a lattice $(L, <,>)$, that is, a free $\mathbf{Z}$-module L with a symmetric bilinear form $<,>: L \times L \to \mathbf{Z}$, the dual lattice L^* is defined as

$$L^* = Hom(L, \mathbf{Z}) = \{x \in L \otimes \mathbf{Q} : < x, y > \in \mathbf{Z} \text{ for all } y \in L\}.$$

The discriminant group of L is L^*/L and the discriminant form of an even lattice (so $< x, x > \in 2\mathbf{Z}$ for all $x \in L$) is the map

$$q : L^*/L \longrightarrow \mathbf{Q}/2\mathbf{Z}, \qquad q(x + L) = < x, x > \mod 2\mathbf{Z}$$

where the bilinear form is extended $\mathbf{Q}$-bilinearly to L^*.

3. Brauer groups of elliptic fibrations on K3 surfaces

3.1. **The transcendental lattice of an elliptic K3 fibration.** A general elliptic K3 surface $f : X \to \mathbf{P}^1$ with a section has Néron-Severi group isometric to U, the hyperbolic plane. It is spanned by the class of a fiber f and the section s. As $f^2 = 0, fs = 1, s^2 = -2$, we get $f(f + s) = 1, (f + s)^2 = 0$ hence f and $f + s$ are a standard basis of U. The elliptic fibration, and the section, on X are unique (cf. 5.4). We will simply write $X/\mathbf{P}^1$ for this unique Jacobian fibration.

There is, up to isometry, a unique embedding $U \hookrightarrow \Lambda_{K3}$ ([N], Thm 1.14.4) hence the transcendental lattice $T_X = NS(X)^\perp$ of X is isometric to:

$$T_X \cong \Gamma := U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

In particular, Γ is unimodular and even.

3.2. **The Brauer group of a general elliptic K3 fibration.** An element α of order b in $Br(X)$ defines a surjective homomorphism $\alpha : \Gamma \to \mathbf{Z}/b\mathbf{Z}$ (cf. 2.1) and determines the sublattice $T_{<\alpha>} = \ker(\alpha : \Gamma \longrightarrow \mathbf{Q}/\mathbf{Z})$ of index b in Γ. As a first step towards understanding the Brauer group we have the following classification.

3.3. **Proposition.** Let $\Gamma' \subset \Gamma$ be a sublattice of index b such that Γ/Γ' is cyclic. Then there exists a $c \in \mathbf{Z}$ such that

$$\Gamma' \cong \Gamma_{b,c} := \Lambda_{b,c} \oplus U \oplus E_8(-1)^2,$$

where $\Lambda_{b,c}$ is the indefinite lattice of rank two defined as:

$$\Lambda_{b,c} = \left\{ \mathbf{Z}^2, (0, b, 2c) := \begin{pmatrix} 0 & b \\ b & 2c \end{pmatrix} \right\}.$$

The lattice $\Gamma_{b,c}$ is determined, up to isometry, by its rank, signature and discriminant form.

Moreover, if $gcd(b, 2c) = 1$ then:
$\Gamma_{b,c} \cong \Gamma_{b,d}$ iff $c \equiv d \bmod b$ and there is an $a \in \mathbf{Z}$ such that $d \equiv a^2 c \bmod b^2$.

Proof. Since Γ is unimodular, there is a $\gamma \in \Gamma$ such that

$$\Gamma' = \ker\Big(\Gamma \longrightarrow \mathbf{Z}/b\mathbf{Z}, \qquad x \longmapsto \langle x, \gamma\rangle \mod b\mathbf{Z}\Big),$$

and we may assume that γ is a primitive vector.

Let $\gamma^2 = -2c$ and assume that $c \neq 0$. As Γ is even, unimodular, of signature $(2, 18)$, the embedding of the sublattice $M = \mathbf{Z}\gamma$ into Γ is unique up to isomorphism ([N], Thm 1.14.4). In particular, we may assume, up to isometries of Γ, that $\gamma = (1, -c) \in U \subset \Gamma$, where U is the first summand of Γ. Then Γ' is generated by $(0, b), (1, c) \in U \subset \Gamma$ and the unimodular summand $U \oplus E_8(-1)^2$. Thus $\Gamma' \cong \Gamma_{b,c}$.

By [N], Cor. 1.13.3, the lattice $\Gamma_{b,c}$ is determined up to isometry by its rank, signature and discriminant form. The discriminant group $A_{b,c}$ of $\Gamma_{b,c}$ is the one of $\Lambda_{b,c}$, which has order $disc(\Lambda_{b,c}) = b^2$. If $gcd(b, 2c) = 1$, $A_{b,c}$ is cyclic of order b^2 with generator $(-2c, b)/b^2$ and the discriminant form q takes the value $q((-2c, b)/b^2) = -2c/b^2 \in \mathbf{Q}/2\mathbf{Z}$. If $\Gamma_{b,d}$ has isomorphic discriminant group, then $gcd(b, 2d) = 1$ (else $A_{b,d}$ is not cyclic) and an isomorphism $A_{b,c} \rightarrow A_{b,d}$ maps the generator $(-2c, b)/b^2$ to the generator $a(-2d, b)/b^2$, for some integer a with $gcd(a, b^2) = 1$. The isomorphism should respect discriminant forms, hence $-2c/b^2 = a^2(-2d/b^2)$ in $\mathbf{Q}/2\mathbf{Z}$, so $c \equiv a^2 d \bmod b^2$. Viceversa, such an $a \in \mathbf{Z}$ defines an isomorphism $A_{b,c} \cong A_{b,d}$.

In case $\gamma^2 = 0$, choose any $\delta' \in \Gamma$ with $\gamma\delta' = 1$, such a δ' exists since γ is primitive and Γ is unimodular. If $(\delta')^2 = 2d$, define $\delta := \delta' - d\gamma$, then the sublattice $H := \mathbf{Z}\gamma + \mathbf{Z}\delta$ is isomorphic to U. Hence $\Gamma \cong H \oplus H^\perp$ where $H^\perp$ is an even unimodular lattice (isomorphic to $U \oplus E_8(-1)^2$). Then Γ' is generated by γ, $b\delta$ and $H^\perp$, so $\Gamma' \cong \Gamma_{b,0}$. $\square$

3.4. **Corollary.** Let $b > 2$ be a prime number. Then there are three isomorphism classes of lattices $\Gamma_{b,c}$: those with $c \equiv 0 \bmod b$, those where c is congruent to a square mod b and those where c is not congruent to a square mod b.

Proof. The condition $c \equiv a^2 d \bmod b^2$ implies of course $c \equiv a^2 d \bmod b$, and it is easy to see that the converse holds as well. $\square$

3.5. **The case** $b = 2$**.** In case $b = 2$ there are two isomorphism classes with representatives $\Gamma_{2,0}$ and $\Gamma_{2,1}$ respectively. As in the proof of Proposition 3.3 let $\alpha \in Br(X)_2 = Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$, correspond to the homomorphism $x \mapsto < x, \gamma >$ defined by $\gamma \in \Gamma$. The quadratic form on Γ induces a degenerate quadratic form:

$$\Gamma/2\Gamma \cong \mathbf{F}_2^{20} \longrightarrow \mathbf{F}_2, \qquad \bar{\gamma} \longmapsto \langle\gamma,\gamma\rangle/2 \mod 2\mathbf{Z}.$$

This is an 'even' quadratic form. In particular, it has $2^{19}(2^{20}+1)$ zeroes in $\mathbf{F}_2^{20}$, including $\bar{\gamma} = 0$. Thus there are $2^{19}(2^{20}+1) - 1$ elements in $Br_2(X)$ with $T_{<\alpha>} \cong \Gamma_{2,0}$ and there are $2^{19}(2^{20}-1)$ elements with $T_{<\alpha>} \cong \Gamma_{2,1}$.

3.6. **Embeddings of $T_{<\alpha>}$ in Λ_{K3}.** Given an $\alpha \in Br(X)$ of order n, the sublattice $T_{<\alpha>}$ of T_X is isomorphic to some $\Gamma_{b,c}$ as in Proposition 3.3. Any lattice $\Gamma_{b,c}$ has a primitive embedding into the K3 lattice. For example, one can simply map the standard basis of $\Lambda_{b,c}$ to $((1,0),(b,0))$, $((0,0),(2c,1)) \in U^2$, the first two copies of U in Λ_{K3}, and extend it by the identity on $U \oplus E_8(-1)^2$ to all of $\Gamma_{b,c}$. In particular, there is a K3 surface Y with $T_Y \cong T_{<\alpha>}$ (cf. 2.4), which is not surprising in view of the Ogg-Shafarevich theory (cf. 1.2).

The orthogonal complement of $i(\Gamma_{b,c})$ depends in general on the primitive embedding i. In fact, we will see that it is isomorphic to $\Lambda_{b,-a^2c}$ for some $a \in \mathbf{Z}$ but the classification of the lattices $\Lambda_{b,d}$ up to isomorphism is different from the one of the $\Gamma_{b,c}$ (cf. Proposition 3.3). Indeed, the summand $U \oplus E_8(-1)^2$ in $\Gamma_{b,c}$ leads to fewer isomorphism classes. Therefore it can happen that $\Gamma_{b,c} \cong \Gamma_{b,d}$ but $\Lambda_{b,c} \not\cong \Lambda_{b,d}$.

3.7. **Proposition.** Let b be a non-zero integer and let $i : \Gamma_{b,c} \hookrightarrow \Lambda_{K3}$ be a primitive embedding. Then there is a non-zero integer a such that

$$i(\Gamma_{b,c})^\perp \cong \Lambda_{b,-a^2c}.$$

Conversely, for any non-zero integer a there is a primitive embedding such that $i(\Gamma_{b,c})^\perp \cong \Lambda_{b,-a^2c}$.

For $b, c, d, n \in \mathbf{Z}$ with $b \neq 0$, $cd \equiv 1 \mod b$ we have:

$$\Lambda_{b,c} \cong \Lambda_{-b,c} \cong \Lambda_{b,c-nb} \cong \Lambda_{b,d}.$$

Moreover, if $gcd(b,c) = 1$ and $0 \leq c, d \leq b-1$ then:
$\Lambda_{b,c} \cong \Lambda_{b,d}$ iff $cd \equiv 1 \mod b$.

Proof. As Λ_{K3} is unimodular, the lattice $i(\Gamma_{b,c})^\perp$ has the same discriminant $-b^2$ as $\Gamma_{b,c}$ and it has the opposite discriminant form. Moreover it has rank two and it is indefinite. Writing the quadratic form on $i(\Gamma_{b,c})^\perp$ as $kx^2 + 2lxy + my^2$, its discriminant is $b^2 = l^2 - km$, hence there is an isotropic vector v in $i(\Gamma_{b,c})^\perp$, i.e. $v \neq 0, v^2 = 0$. We can choose v to be primitive. Thus there is a $w \in i(\Gamma_{b,c})^\perp$ such that v, w is a basis of $i(\Gamma_{b,c})^\perp$ and the quadratic form on this basis has $k = 0$. Then $l = \pm b$, hence $i(\Gamma_{b,c})^\perp \cong \Lambda_{b,d}$ for some d. The discriminant form of $i(\Gamma_{b,c})^\perp$ is the opposite of the one of $\Gamma_{b,c}$ which we determined in the proof of Proposition 3.3 and we get $i(\Gamma_{b,c})^\perp \cong \Lambda_{b,-a^2c}$. For the converse, use [N] Prop. 1.4.1, which shows that there exists an even unimodular lattice containing $\Gamma_{b,c} \oplus \Lambda_{b,-a^2c}$ as sublattice of finite index. The classification of even unimodular lattices implies that this overlattice is isomorphic to Λ_{K3} and gives the desired embedding i.

Let e_1, e_2 be the standard basis of $\Lambda_{b,c}$, so $e_1^2 = 0$, $e_1e_2 = b$ and $e_2^2 = 2c$. Let $f_1 := -e_1, f_2 = e_2$, then $f_i^2 = e_i^2$ and $f_1f_2 = -b$, so $\Lambda_{b,c} \cong \Lambda_{-b,c}$. Next, let $f_1 = e_1, f_2 := -ne_1 + e_2$ then $f_1^2 = 0$, $f_1f_2 = b$, $f_2^2 = 2(c - nb)$, hence $\Lambda_{b,c} \cong \Lambda_{b,c-nb}$. Moreover, if $cd \equiv 1 \bmod b$, then $cd - be = 1$ for an $e \in \mathbf{Z}$ and we have:

$$ {}^tA \begin{pmatrix} 0 & b \\ b & 2c \end{pmatrix} A = \begin{pmatrix} 0 & b \\ b & 2d \end{pmatrix}, \qquad \text{with} \qquad A = A_{b,c,d} = \begin{pmatrix} -c & -e \\ b & d \end{pmatrix} \in GL(2, \mathbf{Z}). $$

The last statement follows by explicitly considering two by two integral invertible matrices A satisfying ${}^tA(0, b, 2c) = (0, b, 2d)A^{-1}$. (cf. [St], Lemma 4.1). $\square$

4. Genus one fibrations on K3 surfaces

4.1. In the previous section we considered a general elliptic fibration $X \to \mathbf{P}^1$ on a K3 surface. We found that for $\alpha \in Br(X)$ of order b the lattice $T_{<\alpha>}$ is isomorphic to the transcendental lattice of a K3 surface Y and that the Néron-Severi group of Y is isomorphic to $\Lambda_{b,c}$ (as in Proposition 3.3), but c is in general not uniquely determined by $T_{<\alpha>}$. We may, and will, assume that $b > 0$ and $0 \leq c < b$ from now on.

From Ogg-Shafarevich theory we know that Y has a genus one fibration $p_Y : Y \to \mathbf{P}^1$ with a multisection of degree b and the relative Jacobian fibration is $X \to \mathbf{P}^1$. We will now see that in general Y has two genus one fibrations and that the relative Jacobians of these fibrations are not always isomorphic.

4.2. **Remark.** If $Z \to \mathbf{P}^1$ is a genus one fibration on a general algebraic K3 surface, then it is easy to see that $NS(Z) \cong \Lambda_{b,c}$ for some $b > 0$.

4.3. **The Kähler cone of $Y_{b,c}$.** To study the geometry (genus one fibrations, automorphisms and projective models) of a K3 surface $Y = Y_{b,c}$ with $NS(Y) \cong \Lambda_{b,c}$ we need to know its Kähler cone $C_{b,c}^+ \subset NS(Y_{b,c})$. For this we need to determine the set $\Delta \subset NS(Y)$ of (classes of) (-2)-curves:

$$ \Delta = \{D \in NS(Y) : \; D > 0, \; D \text{ irreducible}, \; D^2 = -2\} $$

([BPV], Chapter VIII, Cor. 3.8). The following easy lemma supplies this information.

4.4. **Lemma.** Let $Y_{b,c}$ be a K3 surface with Néron-Severi group isomorphic to $\Lambda_{b,c}$, $0 \leq c < b$. Then there is an isomorphism of lattices $NS(Y_{b,c}) \otimes \mathbf{R} \cong \Lambda_{b,c} \otimes \mathbf{R} \cong \mathbf{R}^2$ such that:

(1) if $c \neq b - 1$,

$$ C_{b,c}^+ = \{(x, y) \in \mathbf{R}^2 : \; y > 0 \text{ and } bx + cy > 0\} \qquad (0 \leq c < b - 1), $$

and there are no (-2)-curves on $Y_{b,c}$.

(2) if $c = b - 1$,

$$ C_{b,b-1}^+ = \{v = (x, y) \in \mathbf{R}^2 : v \in C \text{ and } bx + (b - 2)y > 0\} \qquad \text{and} \qquad N = (-1, 1), $$

where N is the unique (-2)-curve on $Y_{b,b-1}$.

Proof. Let $C \subset NS(Y) \otimes \mathbf{R}$ be the connected component of the set $\{v \in NS(Y) \otimes \mathbf{R} : \ v^2 > 0\}$ which contains an ample divisor. Any isometry $NS(Y_{b,c}) \cong \Lambda_{b,c}$ maps C to one of the two connected components of the set $\{(x,y) \in \mathbf{R}^2 : y(bx+cy) > 0\}$. Since $-I$ is an isometry of $\Lambda_{b,c}$, we can choose an isometry $NS(Y) \cong \Lambda_{b,c}$ which identifies

$$C = \{(x,y) \in \mathbf{R}^2 : \ y > 0 \text{ and } bx + cy > 0\}.$$

To determine Δ, note that $v^2 = 2y(bx+cy)$ and $0 \leq c < b$ imply: $v^2 = -2$ iff $c = b-1$ and $v = \pm(1,-1)$. Moreover, as $v^2 = -2$, either v or $-v$ is the class of an effective divisor and this must be the class of a (-2)-curve. As the reflection s_v (defined by $s_v(w) = w + (w,v)v$) is an isometry and $s_v(v) = -v$, we may assume that

$$\Delta = \{(-1,1)\} \ \subset \ \Lambda_{b,b-1}.$$

The Kähler cone C^+ of Y is

$$C^+ = \{v = (x,y) \in \mathbf{R}^2 : v \in C \text{ and } vn > 0 \text{ for all } n \in \Delta\}.$$

([BPV], Chapter VIII, Cor. 3.8). □

4.5. **Automorphisms.** The Torelli theorem for K3 surfaces ([BPV], VIII.11) allows one to determine the automorphism group of a K3 surface. It coincides (via the induced action on $H^2(X,\mathbf{Z})$) with the group of effective Hodge isometries, i.e. Hodge isometries which map the Kähler cone into itself. We will assume in the remainder of this section that the only Hodge isometries of T_X are $\pm I$. This is the case for a general K3 surface with a transcendental lattice of rank at least 3, and thus holds for the general $Y_{b,c}$.

4.6. **Lemma.** Let $d = gcd(b,c)$ and $b' = b/d$, $c' = c/d$. Then the orthogonal group $O(\Lambda_{b,c})$ of the lattice $\Lambda_{b,c}$ is:

$$O(\Lambda_{b,c}) \cong \begin{cases} \{\pm I, \pm J\} & \text{if } c = 0 \text{ or } (c')^2 \equiv 1 \bmod b', \\ \{\pm I\} & \text{else,} \end{cases}$$

where J permutes the two isotropic sublattices of $\Lambda_{b,c}$.

The general K3 surface $Y_{b,c}$ with Néron-Severi group isomorphic to $\Lambda_{b,c}$ has trivial automorphism group except when:

$$(b,c) = (1,0),\ (2,0), \quad \text{or} \quad b > 2,\ c = 1,$$

in which case $Aut(Y_{b,c}) \cong \mathbf{Z}/2\mathbf{Z}$.

Proof. By considering the action of $Aut(\Lambda_{b,c})$ on the primitive isotropic vectors it is clear that this group has at most 4 elements. In case $c^2 \equiv 1 \bmod b$ the map $A_{b,c,c}$ in the proof of Proposition 3.7 gives an isometry, of order two, of the lattice $\Lambda_{b,c}$.

Let $Y = Y_{b,c}$, let T be its transcendental lattice let $\Lambda = \Lambda_{b,c} \cong NS(Y)$ and let $\Lambda^*/\Lambda \cong T^*/T$ be the discriminant group. Any effective Hodge isometry of $H^2(Y,\mathbf{Z})$ must preserve the Kähler cone and, by assumption, it induces $\pm I$ on T. Note that $-I \in Aut(NS(Y))$ does not preserve the Kähler cone.

In case $(b,c) = (1,0)$, $NS(Y)$ is the hyperbolic plane. The subgroup of $Aut(NS(Y))$ preserving the Kähler cone is trivial. As $H^2(Y,\mathbf{Z}) = NS(Y) \oplus T$, there is an effective Hodge isometry on $H^2(Y,\mathbf{Z})$ which induces the identity on the $NS(Y)$ and minus the identity on T and this is the only non-trivial effective Hodge isometry. The corresponding automorphism is the inversion map on the elliptic fibration on $Y = Y_{1,0}$ (cf. 5.4).

In case $b > 1$ and $c = 0$, let $\{I, J\}$ be the subgroup of $Aut(NS(Y))$ preserving the Kähler cone. Then J permutes the basis vectors of $\Lambda^*/\Lambda \cong (\mathbf{Z}/b\mathbf{Z})^2$, whereas -1_T acts as -1 on this group. Hence only in case $b = 2$ there is a non-trivial automorphism on $Y_{b,0}$: it induces the identity on the Néron-Severi group and minus the identity on the transcendental lattice. It is the covering involution for the 2:1 map $Y_{2,0} \to \mathbf{P}^1 \times \mathbf{P}^1$ in 5.5.

The case $0 < c < b$ and $(c')^2 \equiv 1 \bmod b'$. The automorphisms $-I, -A_{b',c',c'}$ of $NS(Y) = \Lambda$ do not preserve the Kähler cone. The vectors $(1/b)(1,0)$, $(1/b)(-c',b') \in \Lambda \otimes \mathbf{Q}$ are in Λ^* and $A_{b',c',c'}$ permutes them. If $A_{b',c',c'}$ lifts to an automorphism of $H^2(Y,\mathbf{Z})$ which is the identity on T, we must have $(1/b)(1,0) \equiv (1/b)(-c',b') \bmod \mathbf{Z}^2$, hence $b' = b$, so $d = 1$ and also $c = c'$. Moreover, as $0 \le c < b$ we get $c = b - 1$. In this case $A_{b,c,c}$ does not preserve the Kähler cone. If $A_{b',c',c'}$ lifts to an automorphism of $H^2(Y,\mathbf{Z})$ which is minus the identity on T, we must have $-(1/b)(1,0) \equiv (1/b)(-c',b') \bmod \mathbf{Z}^2$, hence $b' = b, c' = c$ as before and $c' = 1$. In this case Y has indeed an automorphism of order two which is the covering involution for the 2:1 map $\phi_h : Y \to \mathbf{P}^2$ given by $h = (0,1) \in C^+$ with $h^2 = 2$, cf. the examples 5.6, 5.8 below. □

4.7. **The elliptic fibrations on $Y_{b,c}$.** The elliptic fibrations on $Y_{b,c}$ correspond to the primitive isotropic classes in the closure of the Kähler cone. Thus there are two such fibrations if $c \neq b-1$:

$$\epsilon_1 = (1,0), \quad \epsilon_2 = (-c',b') \quad \in C^+_{b,c}, \qquad \epsilon_1^2 = \epsilon_2^2 = 0,$$

with $b' = b/d, c' = c/d$ and $d = gcd(b,c)$ and we put $b' = 1, c' = 0$ if $c = 0$. In case $c = b - 1$ there is a unique genus one fibration with class $\epsilon_1 = (1,0)$.

In case $b > 2, c = 1$, the non-trivial automorphism of $Y_{b,c}$ interchanges the two elliptic fibrations. More generally, if $Y_{b,c}$ has two elliptic fibrations it is interesting to compare the associated relative Jacobians.

4.8. **Proposition.** Let $b \in \mathbf{Z}_{>0}$ and $0 \le c < b - 1$ and assume that $gcd(b,2c) = 1$. Then the relative Jacobians of the two elliptic fibrations on $Y_{b,c}$ are isomorphic.

In case $(b,c) = (2,0)$ the relative Jacobians are not isomorphic in general.

Proof. If $gcd(b,2c) = 1$, the discriminant group of the transcendental lattice $T \cong \Gamma_{b,c}$ of $Y_{b,c}$ is cyclic of order b^2. Hence it has a unique subgroup H of order b, which is maximal isotropic. Thus $\Gamma_{b,c}$ has a unique even, unimodular overlattice and this overlattice must be $\Gamma_{1,0} \cong U^2 \oplus E_8(-1)^2$. Hence there is a unique Hodge structure on Λ_{K3} such that T is a sublattice of index b in the transcendental lattice of this Hodge structure. As this Hodge structure determines a unique elliptic fibration, this must be the relative Jacobian fibration (for any genus one fibration on X).

The surface $Y_{2,0}$ is a 2:1 cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over a curve of bidegree $(4,4)$ (cf. section 5.5). If the Jacobian fibrations associated to the two elliptic pencils were isomorphic, there would be an automorphism of $\mathbf{P}^1$ carrying the discriminant locus of one fibration into the one of the other fibration, which is easily seen not to be true in general. (In this case there are two maximal isotropic subgroups in the discriminant group $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ of T and the induced Hodge structures on the two overlattices are not necessarily Hodge isometric). □

4.9. **Example.** Let $b>2$ be a prime number. Given $\alpha\in Br(X)$ of order b, the subgroup $<\alpha>$ has $b-1$ elements of order b. Since α and $-\alpha$ define the same genus one fibration, there are $(b-1)/2$ genus one fibrations $Y\rightarrow\mathbf{P}^1$ with $T_Y\cong T_{<\alpha>}$ (so these surfaces are Fourier-Mukai partners) and with associated relative Jacobian $X/\mathbf{P}^1$.

We consider the case $T_{<\alpha>}\cong\Gamma_{b,1}$ with $b\equiv 1 \bmod 4$, so that -1 is a square mod b, in some more detail. Proposition 3.7 shows that the Néron-Severi group isomorphic to $\Lambda_{b,-1}$, $\Lambda_{b,1}$ or $\Lambda_{b,c^2}\cong\Lambda_{b,c^{-2}}$ with $c^2\not\equiv c^{-2} \bmod b$, hence we get $1+1+(b-5)/4$ isomorphism classes. If $NS(Y)\cong\Lambda_{b,-1}$ there is a unique genus one fibration on Y. If $NS(Y)\cong\Lambda_{b,1}$ there are two genus one fibrations on Y, the automorphism of the surface Y permutes them, so they are isomorphic. The other surfaces have two genus one fibrations. These fibrations are not isomorphic in general but both have relative Jacobian fibration $X/\mathbf{P}^1$, so we get $1+1+2((b-5)/4)=(b-1)/2$ genus one fibrations associated to $<\alpha>$ as expected. We conclude that there are $1+1+(b-5)/4=(b+3)/4$ isomorphism classes of K3 surfaces Y with $T_Y\cong T_{<\alpha>}$.

5. Some projective models of K3 surfaces with genus one fibrations

5.1. The well-developed theory of linear series on K3 surfaces (cf. [SD]) allows one to construct projective models for a K3 surface $Y_{b,c}$ with Néron-Severi group isomorphic to $\Lambda_{b,c}$ as in 4.3. For small b we recover some well-known examples.

5.2. **The case $c=b-1$.** When $c=b-1$ and $b\geq 3$, the class $h_0=(0,1)\in C^+_{b,c}$, with $h_0^2=2c$, defines a map:

$$\phi_{h_0}: Y_{b,c}\longrightarrow\mathbf{P}^{c+1}.$$

The unique genus one fibration ϵ_1 has class $(1,0)$. As $h_0\epsilon_1=b$ the curves from this genus one pencil are mapped to degree $b=c+1$ elliptic normal curves. Each curve spans a $\mathbf{P}^c\subset\mathbf{P}^{c+1}$, so it spans a hyperplane section. The residual intersection is $h_0-\epsilon_1=N$, which is the (-2)-curve. As $h_0N=2c-b=c-1$, $\phi_{h_0}(N)$ is a rational normal curve in $\mathbf{P}^{c-1}$ and it meets each elliptic curve from ϵ_1 in $N\epsilon_1=c+2$ points. See also the examples 5.6 and 5.9.

5.3. **The case $c=0$.** When $c=0$, the class $h=(1,1)$ maps $Y=Y_{b,0}$ to $\mathbf{P}^{b+1}$. As $h=\epsilon_1+\epsilon_2$, the sum of the genus one fibrations, if we fix a curve $E_1\in|\epsilon_1|$, the residual intersection of any hyperplane containing E_1 is a curve from $|\epsilon_2|$. As $h\epsilon_1=b$, the curve E_1 is an elliptic curve of degree b which spans a $\mathbf{P}^{b-1}$ in $\mathbf{P}^{b+1}$, so we get a pencil of hyperplanes ($\mathbf{P}^b$'s) whose residual intersections give the system $|\epsilon_2|$. In particular, fixing independent sections $s_1,s_2\in H^0(Y,\epsilon_1)$, $t_1,t_2\in H^0(Y,\epsilon_2)$ we get the sections $z_{ij}=s_it_j\in H^0(Y,h)$ which satisfy $z_{11}z_{22}=z_{12}z_{21}$, hence the image of Y under ϕ_h is contained in a rank 4 quadric whose rulings cut out the linear systems $|\epsilon_1|$ and $|\epsilon_2|$. See the examples 5.5, 5.7.

5.4. $(b, c) = (1, 0)$**.** There is a unique -2-curve N on $Y_{1,0}$ with class $(-1, 1)$ and there is a unique genus one fibration F with class $(1, 0)$. Since $NF = 1$, the curve N is a section of the fibration, so $Y_{1,0}$ has an elliptic (Jacobian) fibration.

Note that if $D \in C^+_{1,0}$ is very ample, then we must have $DF \geq 3$ (else the fibers are not embedded) so the class of D is (a, b) with $a > b \geq 3$ and thus $D^2 \geq 12$.

5.5. $(b, c) = (2, 0)$**.** The two elliptic pencils give a 2:1 map from $Y_{2,0}$ to $\mathbf{P}^1 \times \mathbf{P}^1$ branched over a curve of bidegree $(4, 4)$. The Segre map embeds $\mathbf{P}^1 \times \mathbf{P}^1$ as a smooth quadric in $\mathbf{P}^3$ (cf. 5.3).

5.6. $(b, c) = (2, 1)$**.** Let $D = \epsilon_1 + N = (0, 1)$, so $D^2 = 2$. The 2:1 map $\phi_D : Y \to \mathbf{P}^2$ contracts the (-2)-curve $N = (-1, 1)$, the inverse images of the lines through $\phi_D(N)$ are the union of curves in the genus one fibration and N. Thus $Y_{2,1}$ is obtained as the desingularization of the double cover of $\mathbf{P}^2$ branched along a sextic with one node. In particular, $Y_{2,1}$ is a double cover of the ruled surface $\mathbf{F}_1$.

5.7. $(b, c) = (3, 0)$**.** The K3 surface $Y_{3,0}$ is a complete intersection of a rank 4 quadric and a cubic hypersurface in $\mathbf{P}^4$ (cf. 5.3). The rank 4 quadric has two rulings, i.e. families of $\mathbf{P}^2$'s parametrized by a $\mathbf{P}^1$, each such $\mathbf{P}^2$ intersects the surface along the curve where it intersects the cubic hypersurface.

5.8. $(b, c) = (3, 1)$**.** A general K3 surface Y of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^2$ has $NS(Y) \cong \Lambda_{3,2}$. Let $x = \{pt\} \times \mathbf{P}^2$, $y = \mathbf{P}^1 \times \mathbf{P}^1$ in $Pic(\mathbf{P}^1 \times \mathbf{P}^2)$, then $x^2 = 0, xy = \{pt\} \times \mathbf{P}^1, y^2 = \mathbf{P}^1 \times \{pt\}$ and $xy^2 = 1, y^3 = 0$. Restricting these classes to the K3 surface one finds the lattice $\Lambda_{3,1}$ (for example, y^2 restricts to $y^2(2x + 3y) = 2$). This surface has an obvious elliptic fibration (projection onto $\mathbf{P}^1$) and is also a double cover of $\mathbf{P}^2$ (projection on $\mathbf{P}^2$). Writing the equation of the surface as

$$Y : \quad s^2F + 2stG + t^2H = 0 \qquad (\subset \mathbf{P}^1 \times \mathbf{P}^2)$$

with homogeneous polynomials $F, G, H \in \mathbf{C}[x_0, x_1, x_2]$ of degree 3, the branch curve of the double cover is $G^2 - FH$. In particular, this sextic has tangent cubics (like $F = 0$, in fact for any s, t, the cubic with equation as above is tangent to the branch curve) and these are the fibers of the elliptic fibration. This system of tangent cubics defines an (even, effective) theta characteristic on the branch curve. It has a second elliptic fibration given by $-x + 3y$ which is the image of the original fibration x under the covering involution. The Segre map $\mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^5$ embeds this surface in $\mathbf{P}^5$.

5.9. $(b, c) = (3, 2)$**.** A general quartic surface Y with a line in $\mathbf{P}^3$ has $NS(Y) \cong \Lambda_{3,2}$. The unique genus one fibration is given by the pencil of planes on the line. The general complete intersection of type (1,1) and (1,3) in $\mathbf{P}^1 \times \mathbf{P}^3$ is also of this type. If $x = \{pt\} \times \mathbf{P}^3$ and $y = \mathbf{P}^1 \times \mathbf{P}^2$ then in degree 4 the only non-zero intersection is $xy^3 = 1$. The class of the surface is $(x + y)(x + 3y) = x^2 + 4xy + 3y^2$. Hence restricting x, y to the surface one finds the lattice $\Lambda_{3,2}$. The Segre map

$$\phi_h : Y \longrightarrow \mathbf{P}^1 \times \mathbf{P}^3 \hookrightarrow \mathbf{P}^6 \subset \mathbf{P}^7,$$

is given by $h = x + y$. The equation of type $(1, 1)$ is a linear equation for the image of Y in the Segre embedding, hence the image lies in a hyperplane $\mathbf{P}^6$ in the Segre space $\mathbf{P}^7$.

5.10. $(b, c) = (4, 2)$**.** The general complete intersection Y of bidegree (1,2) and (1,2) in $\mathbf{P}^1 \times \mathbf{P}^3$ has $NS(Y) \cong \Lambda_{4,2}$. If $x = \{pt\} \times \mathbf{P}^3$ and $y = \mathbf{P}^1 \times \mathbf{P}^2$ then in degree 4 the only non-zero intersection is $xy^3 = 1$. The class of the surface is $(x + 2y)^2 = x^2 + 4xy + 4y^2$, hence restricting x, y to the surface and finds the lattice $\Lambda_{4,2}$. The Segre map is $\phi_h : Y \to \mathbf{P}^7$ with $h = x + y$. Using the coordinates $(\lambda : \mu)$ on $\mathbf{P}^1$, the two equations for the surface are:

$$\lambda Q_1 - \mu Q_2, \qquad \lambda Q_3 - \mu Q_4,$$

for homogeneous degree two polynomials $Q_1, \dots, Q_4$ in the coordinates of $\mathbf{P}^3$. In particular, the projection onto $\mathbf{P}^3$ has equation $Q_1Q_4 - Q_2Q_3 = 0$ which is a K3 surface containing the elliptic curves defined by $Q_1 = Q_3 = 0$ and $Q_1 = Q_2 = 0$. These quartic genus one curves lie in distinct pencils.

6. Two torsion in the Brauer group of an elliptic surface

6.1. **Ruled surfaces.** We denote by $\bar{f} : \mathbf{F}_e \to \mathbf{P}^1$ the ruled surface which is the compactification of the line bundle $\mathcal{O}(e)$ on $\mathbf{P}^1$ with a rational curve C_∞ at infinity, with self-intersection $C_\infty^2 = -e$. This surface can be obtained, for $e \geq 1$, as the blow up of the cone in $\mathbf{P}^{e+1}$ over the rational normal curve of degree e in the vertex, in case $e = 0$ we have $\mathbf{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$.

Let F be a fiber of $\bar{f}$, then

$$Pic(\mathbf{F}_e) = \mathbf{Z}F \oplus \mathbf{Z}C_\infty, \qquad F^2 = 0, \quad FC_\infty = 1, \quad C_\infty^2 = -e.$$

The image in $\mathbf{F}_e = \mathcal{O}(e) \coprod C_\infty$ of a global section of $\mathcal{O}(e)$ is a section C of $\mathbf{F}_e \to \mathbf{P}^1$. It is also a hyperplane section of the cone in $\mathbf{P}^{e+1}$ which doesn't meet the vertex. The classes of the canonical line bundle K on $\mathbf{F}_e$ and the class of a section C of $\bar{f}$ are:

$$K = K_{\mathbf{F}_e} = -(e + 2)F - 2C_\infty, \qquad C = eF + C_\infty$$

(K can easily be found using the adjunction formula and the fact that F and C_∞ have genus zero, the class of C can be found using $CF = 1$ and $CC_\infty = 0$).

6.2. **Weierstrass models of elliptic surfaces.** Let $f : X \to \mathbf{P}^1$ be a general elliptic fibration (with section). Then X is the double cover of the ruled surface $\bar{f} : \mathbf{F}_{2d} \to \mathbf{P}^1$ for some $d \in \mathbf{Z}_{\geq 0}$ ([CD] §5.5, [FM] 1.4.1, [Mi]). The branch locus of $\pi : X \to \mathbf{F}_{2d}$ consists of the section at infinity C_∞ of $\bar{f}$ and a curve $\mathcal{C}$ such that $\pi : \mathcal{C} \to \mathbf{P}^1$ is a 3:1 covering of $\mathbf{P}^1$. The topological Euler characteristic of X is given by the Hurwitz formula:

$$\chi(X) = 2\chi(\mathbf{F}_{2n}) - \chi(\mathcal{C} \coprod C_\infty) = 8 - (2 - 2g + 2)) = 4 + 2g.$$

Since $H^1(X, \mathbf{Z}) = H^3(X, \mathbf{Z}) = 0$, the rank of $H^2(X, \mathbf{Z})$ is $2 + 2g$.

We assume from now on that $Pic(X) \cong \mathbf{Z}^2$, so we get

$$Br(X) \cong H^2(X, \mathbf{Q}/\mathbf{Z})/(Pic(X) \otimes (\mathbf{Q}/\mathbf{Z})) \cong (\mathbf{Q}/\mathbf{Z})^{2g}$$

and $\mathcal{C}$ is smooth and irreducible. From the intersection numbers $C_\infty \mathcal{C} = 0$, $F\mathcal{C} = 3$ one finds that $\mathcal{C}$ has class $3C$. The canonical bundle $K_\mathcal{C}$ on $\mathcal{C}$ is the restriction of $K + \mathcal{C}$ to $\mathcal{C}$, that is of $(4d - 2)F + C_\infty$ to $\mathcal{C}$, this gives:

$$\mathcal{C} = 3C = 6dF + 3C_\infty, \qquad g(\mathcal{C}) = 6d - 2.$$

6.3. **The theta characteristic on $\mathcal{C}$.** As the canonical bundle on $\mathcal{C}$ is the restriction of $K+\mathcal{C}=(4d-2)F+C_\infty$ to $\mathcal{C}$ and C_∞ and $\mathcal{C}$ are disjoint, the canonical bundle of $\mathcal{C}$ is also the restriction of $(4d-2)F$ to $\mathcal{C}$. Let θ be the restriction of the divisor class $(2d-1)F$ to $\mathcal{C}$:

$$\theta=(2d-1)F_{|\mathcal{C}},\qquad \text{then}\qquad 2\theta=K_{\mathcal{C}}$$

hence θ is a theta characteristic on $\mathcal{C}$.

6.4. **Lemma.** The theta characteristic θ on $\mathcal{C}$ is even and $\dim H^0(\mathcal{C},\theta)=2d$.

Proof. Let $D=\mathcal{C}\cap F$, a divisor of degree three on $\mathcal{C}$ with $h^0(D)=2$, let $1,x$ be a basis of $H^0(D)$ (so the map $\bar{f}:\mathcal{C}\to\mathbf{P}^1$ is given by these two functions). As $\theta=(2d-1)D$, we get $1,x,\ldots,x^{2d-1}\in H^0(\theta)$.

For $y\in H^0(\theta)$, we get $y,xy,\ldots x^{2d-1}y\in H^0(2\theta)=H^0(K_{\mathcal{C}})$, this space also contains $1,x,\ldots,x^{4d-2}$. Hence if y is not a linear combination of powers of x, then $\dim H^0(K_{\mathcal{C}})\geq 2d+4d-1=6d-1$, but $\dim H^0(K_{\mathcal{C}})=g(\mathcal{C})=6d-2$, so we get a contradiction. □

6.5. **Remark.** The theta characteristic θ on $\mathcal{C}$ gives a natural bijection between the points of order two in $Pic(\mathcal{C})$ and the theta characteristics of $\mathcal{C}$ given by $\alpha\mapsto\alpha+\theta$.

6.6. We will show that in case $Pic(X)\cong\mathbf{Z}^2$ the group $Br(X)_2$ is isomorphic to $Br(\mathbf{F}_{2d}-\mathcal{C})$ via pull-back of Azumaya algebras along the cover $X\to\mathbf{F}_{2d}$ and extension over the ramification curve in X. We will describe the corresponding Azumaya algebras in the next section.

6.7. **An exact sequence.** Artin and Mumford ([AM] Theorem 1) give an exact sequence which relates the Brauer group of a smooth projective surface S over an algebraically closed field k with $H^1_{\text{ét}}(S,\mathbf{Q}/\mathbf{Z})=0$ to the Brauer group of its function field $k(S)$. We will work over $k=\mathbf{C}$ in order to simplify the exposition.

Let $\{C_1,\ldots,C_r\}$ be a finite set of distinct curves in a smooth surface S which intersect in points $\{p_1,\ldots,p_s\}$. Let

$$U=S-(C_1\cup\ldots\cup C_r),\qquad T_i=C_i\cap(\cup_{j\neq i}C_j\}\subset\{p_1,\ldots,p_s\}.$$

An Azumaya algebra over U defines a class D in $Br(k(S))$, the Brauer group of the field $k(S)$ of rational functions on S.

The cohomology group

$$H^1_{\text{ét}}(k(C_i),\mathbf{Q}/\mathbf{Z})\cong Hom(Gal(\overline{k(C_i)}/k(C_i),\mathbf{Q}/\mathbf{Z})$$

is the group of cyclic extensions of the field of rational functions $k(C)$. We denote by $H^1(k(C),\mathbf{Q}/\mathbf{Z})_T$ the subgroup of cyclic extensions unramified outside $T\subset C$. In particular, if C is a smooth curve of genus g and $\sharp T=m$, then

$$H^1(k(C),\mathbf{Q}/\mathbf{Z})_T\cong Hom(\pi_1(C-T),\mathbf{Q}/\mathbf{Z})\cong(\mathbf{Q}/\mathbf{Z})^{2g+m-1}\qquad\text{unless } m=0,$$

in which case $H^1(k(C),\mathbf{Q}/\mathbf{Z})_T\cong(\mathbf{Q}/\mathbf{Z})^{2g}$.

Theorem 1 from [AM] shows that an Azumaya algebra over U extends to S iff the image of its generic fiber under the map $a_i : Br(k(S)) \to H^1(k(C_i), \mathbf{Q}/\mathbf{Z})$ is trivial for each i. Moreover, this theorem provides an exact sequence:

$$0 \to Br(U)/Br(S) \xrightarrow{a} \oplus_{i=1}^r H^1(k(C_i), \mathbf{Q}/\mathbf{Z})_{T_i} \xrightarrow{r} \oplus_{j=1}^s \mu(-1) \xrightarrow{s} \mu(-1) \to 0,$$

where the map $r = (r_j)$ measures the ramification in p_j of the cyclic coverings of the C_i. Here μ_n is the group of n-th roots of unity, $\mu = \cup_n \mu_n$, $\mu(-1) = Hom(\mu, \mathbf{Q}/\mathbf{Z})$ ($\cong \mathbf{Q}/\mathbf{Z}$) and the map s is the sum map.

6.8. **Theorem.** Let $\pi : X \to \mathbf{F}_{2d}$ be the Weierstrass model of an elliptic fibration with branch curve $\mathcal{C} \coprod C_\infty$, and $NS(X) \cong \mathbf{Z}^2$. In particular, $\mathcal{C}$ is a smooth curve of genus $g = 6d - 2$.

Then there is a natural isomorphism:

$$Br(X)_2 \cong J(\mathcal{C})_2.$$

Proof. We will show that $Br(U') \cong J(\mathcal{C})_2$, where

$$U' = \mathbf{F}_{2d} - \mathcal{C},$$

and that $\pi^* : Br(U') \to Br(f^{-1}(U'))$ is injective.

The theorem will follow from this. Indeed, as $\mathbf{F}_{2d}$ is a rational surface, we have $H^2(\mathbf{F}_{2d}, \mathcal{O}) = 0$ and thus $Br(\mathbf{F}_{2d}) = 0$. Therefore the exact sequence in 6.7 (with $r = 1$ and T_1 the empty set) gives an isomorphism

$$a_{U'} : Br(U') \xrightarrow{\cong} J(\mathcal{C})_2.$$

Geometrically, the conic bundle on U' defined by an element of order two in $Br(U')$ extends to a conic bundle on S which ramifies over the smooth curve $\mathcal{C}$, so the inverse image of each point of $\mathcal{C}$ is the union of two lines. This provides an unramified double cover of $\mathcal{C}$, or, equivalently, a point of order two on $J(\mathcal{C})$ (cf. [B1]). The pull-back along f gives an Azumaya algebra on $f^{-1}(U')$. Due to the branching of order two along $\mathcal{C} = \mathbf{F}_{2d} - U'$, this Azumaya algebra extends to all of X (locally near a point of X on the ramification locus, the conic bundle is defined by a homogeneous polynomial of the form $X^2+Y^2+t^2Z^2$, where $t = 0$ defines the ramification curve. Changing coordinates $Z := tZ$ shows that the conic bundle does not ramify in codimension one, hence does not ramify at all). We will show that $\pi^* : Br(U') \to Br(X)$ is injective, but both groups have 2^{2g} elements (cf. 6.2), so π^* is an isomorphism.

It remains to show that $\pi^* : Br(U') \to Br(\pi^{-1}(U'))$ is injective. We use the methods of [KPS] and [F]. For this we need to work over an affine open subset U of $\mathbf{F}_{2d}$ (and its inverse image V in X). The surface $\mathbf{F}_{2d}$ maps onto the cone in $\mathbf{P}^{2d+1}$ over a rational normal curve of degree $2d$ in $\mathbf{P}^{2d}$. This map contracts the divisor C_∞ to the vertex of the cone and maps the fibers of the fibration to lines. The sections of the fibration map to (linear) hyperplane sections which do not pass through the vertex of the cone. For any point on $\mathbf{P}^1$ there is a hyperplane section which consists of the fiber over that point (with multiplicity $2d$) and there is a cubic hypersurface in $\mathbf{P}^{2d+1}$ which cuts out $\mathcal{C}$. The inverse image of these sections in $\mathbf{F}_{2d}$ is the union of three curves, $C_\infty, F, \mathcal{C}$, where F now stands for a specific fiber. Thus

$$U = \mathbf{F}_{2d} - \{C_\infty, F, \mathcal{C}\}$$

is an affine open subset since it is isomorphic to the complement of a (degree four) hyperplane section of a projective surface in $\mathbf{P}^{2d+1}$.

As the smooth rational curve C_∞ meets $F \cup \mathcal{C}$ in just one point in $C_\infty \cap F$ we get that $H^1(k(C_\infty), \mathbf{Q}/\mathbf{Z})_{C_\infty \cap (F \cup \mathcal{C})}$ is trivial. Moreover, F meets $\mathcal{C}$ in three distinct points (in general), F is a smooth rational curve and $\mathcal{C}$ is a smooth curve of genus g, so we get the exact sequence:

$$0 \to Br(U) \xrightarrow{a} (\mathbf{Q}/\mathbf{Z})^3 \oplus (\mathbf{Q}/\mathbf{Z})^{2g+2} \xrightarrow{r} \mu(-1)^4 \xrightarrow{s} \mu(-1) \to 0.$$

From this we find that

$$Br(U)_2 \cong (\mathbf{Z}/2\mathbf{Z})^{2g+2}.$$

Next we can compute $Br(V)_2$ in the same way as $Br(U)_2$. The double cover $X \to \mathbf{F}_{2d}$ branches over C_∞ and $\mathcal{C}$ and the inverse image of F is a smooth elliptic curve E, which meets (the inverse images of) C_∞ and $\mathcal{C}$ in one respectively three points. This gives the exact sequence:

$$0 \to Br(V)/Br(X) \xrightarrow{a} (\mathbf{Q}/\mathbf{Z})^5 \oplus (\mathbf{Q}/\mathbf{Z})^{2g+2} \xrightarrow{r} \mu(-1)^4 \xrightarrow{s} \mu(-1) \to 0.$$

Since $Br(X) \cong (\mathbf{Q}/\mathbf{Z})^{2g}$ we get

$$Br(V)_2 \cong (\mathbf{Z}/2\mathbf{Z})^{4g+4}.$$

This sequence also shows that $Br(V')_2 \cong (\mathbf{Z}/2\mathbf{Z})^{4g+2}$ where $V' \subset X$ is the inverse image of U'.

Now we show that π^*, also denoted by res, induces an injective map, $Br(U)_2 \to Br(V)_2$. From [KPS], Theorem 6.1 we have an exact sequence:

$$Br(U)_2 \xrightarrow{res} \ker\left(Br(V)_2 \xrightarrow{cores} Br(U)_2\right) \longrightarrow Pic(U)/cores(Pic(V)).$$

We will not need to know the definition of $cores$. Note that $Pic(U) = 0$ since $U \subset \mathbf{F}_{2d} - \{C_\infty, F\}$ and $Pic(\mathbf{F}_{2d}) = \mathbf{Z}C_\infty \oplus \mathbf{Z}F$. As $Br(U)_2 \cong (\mathbf{Z}/2\mathbf{Z})^{2g+2}$ and $Br(V)_2 \cong (\mathbf{Z}/2\mathbf{Z})^{2(2g+2)}$ we must have that $res : Br(U)_2 \longrightarrow Br(V)_2$ is injective (and we also get that $\mathrm{im}(res) = \ker(cores)$). In particular, the subgroup $Br(U')_2$ of $Br(U)_2$ injects into $Br(V)_2$. As these algebras are unramified over the curves in $U' - U$, they are also unramified over the curves in $V' - V$ hence $Br(U')_2 \hookrightarrow Br(V')_2$ as desired. □

7. Conic bundles over Jacobian fibrations

7.1. In this section we consider a genus one fibration $g : Y \to \mathbf{P}^1$ with a double section and we recover the associated Jacobian fibration $f : X \to \mathbf{P}^1$. For this we only use the classical construction of the Jacobian of a double cover of $\mathbf{P}^1$ branched over four points, due to Hermite (cf. [We], [Mu3]), i.e. we consider the generic fibers of g and f over the field $k(\mathbf{P}^1)$. This construction also gives us a conic bundle over X which corresponds to $\alpha \in Br(X)$ defining Y.

7.2. **The Jacobian fibration.** Let $g : Y \to \mathbf{P}^1$ be a a genus one fibration defined by an equation

$$Y : \qquad w^2 = a_0v^4 + 4a_1v^3 + 6a_2v^2 + 4a_3v + a_4$$

The Jacobian fibration of g was found by Hermite (cf. [We]). We will follow [Mu3], 1.3b and 11.3c. The Jacobian fibration of $g : Y \to \mathbf{P}^1$ as above is the elliptic surface $f : X \to \mathbf{P}^1$ given by

$$X : \qquad y^2 = 4x^3 - g_2x - g_3,$$

with

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$

$$g_3 = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3.$$

In case Y is a K3 surface, the a_i are homogeneous polynomials and the g_2 and g_3 are homogeneous of degree 8 and 12 respectively (cf. 7.4 and 7.5), so the K3 surface X maps onto $\mathbf{F}_4$.

We will assume that the curve

$$\mathcal{C}: \qquad 4x^3 - g_2x - g_3 = 0 \qquad (\subset \mathbf{F}_{2d})$$

in the ruled surface $\mathbf{F}_{2d}$ is smooth. The surface X is a double cover of $\mathbf{F}_{2d}$, the branch curve in $\mathbf{F}_{2d}$ is the union of $\mathcal{C}$ and the 'section at infinity' C_∞.

7.3. The explicit expressions for g_2 and g_3 imply that the equation of $\mathcal{C}$ is a symmetric determinant:

$$4x^3 - g_2x - g_3 = \det(M), \qquad M = \begin{pmatrix} a_0 & a_1 & a_2 + 2x \\ a_1 & a_2 - x & a_3 \\ a_2 + 2x & a_3 & a_4 \end{pmatrix}.$$

Thus M defines a conic bundle on $\mathbf{F}_{2d}$ which degenerates over the curve $\mathcal{C}$. In particular, it defines an étale 2:1 cover of $\mathcal{C}$ (the conics split in two lines over points in $\mathcal{C}$), thus we get a point α of order two in $Pic(\mathcal{C})$ (cf. [B1]). From sections 3.5, 5.5, 5.6 it follows that the theta characteristic $\alpha + \theta$, where θ is the even theta characteristic on $\mathcal{C}$ defined in 6.3, is odd in the case of a double $\mathbf{F}_1$ and even in the case of a double quadric.

Alternatively, one could try to define α as the cokernel of a map of locally free sheaves $M : \mathcal{G} \to \mathcal{F}$ (as in the case of symmetric matrices with entries in $\mathbf{C}[x_0, x_1, x_2]$ where one considers bundles over $\mathbf{P}^2$, cf. [B2], [BCZ]).

7.4. **Double quadrics.** (See 5.5.) Let $G \in \mathbf{C}[s, t, u, v]$ be bihomogeneous of degree 4 in s, t and in u, v, so $G(\lambda s, \lambda t, \mu u, \mu v) = \lambda^4\mu^4 G(s, t, u, v)$. Then G defines a curve $\mathcal{R}$ of bidegree $(4, 4)$ and genus 9 in $\mathbf{P}^1 \times \mathbf{P}^1 \cong \mathbf{F}_0$. Let $\bar{g} : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ be the projection to the first factor. The double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over $\mathcal{R}$ is a K3 surface Y with a genus one fibration $g : Y \to \mathbf{P}^1$, the elliptic curves are the inverse images of the fibers of $\bar{g}$. This fibration can be defined by an equation of the form:

$$w^2 = a_0v^4 + a_1v^3 + a_2v^2 + a_3v + a_4, \qquad a_i \in \mathbf{C}[s, t], \qquad deg(a_i) = 4,$$

where $G(s, t, u, v) = a_0(s, t)v^4 + \ldots + a_4(s, t)u^4$. Note that each a_i is homogeneous of degree 4.

7.5. **Nodal sextics.** (See 5.6.) Let $F \in \mathbf{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree six which defines a plane sextic $F = 0$ with a node in $(0 : 0 : 1)$. Blowing up $\mathbf{P}^2$ in this point, one obtains the ruled surface $\mathbf{F}_1 \subset \mathbf{P}^1 \times \mathbf{P}^2$, the map $\bar{g} : \mathbf{F}_1 \to \mathbf{P}^1$ is the projection on the first factor. The double cover of $\mathbf{F}_1$ branched over the strict transform $\mathcal{R}$ of the sextic (which has genus $10 - 1 = 9$) is a K3 surface Y with a genus one fibration $g : Y \to \mathbf{P}^1$, the elliptic curves are the inverse images of the fibers of $\bar{g}$. This fibration is defined by an equation of the form:

$$w^2 = a_0v^4 + a_1v^3 + a_2v^2 + a_3v + a_4, \qquad a_i \in \mathbf{C}[s, t], \qquad deg(a_i) = i + 2,$$

where $F(x_0, x_1, x_2) = a_0(x_0, x_1)x_2^4 + \ldots + a_4(x_0, x_1)$. Note that a_i is homogeneous of degree $i + 2$.

7.6. **The trigonal construction.** In the two cases above, the branch curve $\mathcal{R} \subset \mathbf{F}_e$, $e = 0, 1$, of $Y \to \mathbf{F}_e$ is a curve of genus 9. The fibration $\mathbf{F}_e \to \mathbf{P}^1$ induces a 4:1 map $\mathcal{R} \to \mathbf{P}^1$, i.e. $\mathcal{R}$ is a tetragonal curve. The genus one fibration on Y corresponds to a point of order two $\alpha \in Br(X)_2 \cong J(\mathcal{C})_2$, where $X \to \mathbf{P}^1$ is the relative Jacobian of $Y/\mathbf{P}^1$ and $\mathcal{C}$ is the branch curve of $X \to \mathbf{F}_4$. The fibration $\mathbf{F}_4 \to \mathbf{P}^1$ induces a 3:1 map $\mathcal{C} \to \mathbf{P}^1$, so we get a trigonal curve $\mathcal{C}$ with a point of order two $\alpha \in J(\mathcal{C})_2$. The pairs $(\mathcal{C}, \alpha)$ correspond to étale 2:1 covers $\tilde{\mathcal{C}}_\alpha \to \mathcal{C}$.

The relation between $(\mathcal{C}, \alpha)$ and $\mathcal{R}$ is given by Recillas' trigonal construction ([R], [Don] §2.4) which gives a natural bijection between étale double covers of trigonal curves and tetragonal curves. In fact the trigonal construction associates to a 4:1 cover of $\mathbf{P}^1$, that is to an extension of degree four of $k(\mathbf{P}^1)$, the cubic extension of $k(\mathbf{P}^1)$ given by the Lagrange resolvent and its natural quadratic extension. The explicit formula for the Lagrange resolvent (cf. [Go], II, §174) shows that the cubic extension corresponds to the 3:1 cover $\mathcal{C} \to \mathbf{P}^1$.

7.7. **An Azumaya algebra.** The symmetric matrix M in 7.3 defines a conic bundle on $\mathbf{F}_{2d}$ which ramifies over $\mathcal{C}$, its pull-back to $X - \mathcal{C}$ extends to an unramified conic bundle over all of X (cf. the proof of Theorem 6.8). The associated even Clifford bundle is an Azumaya algebra, actually it is a bundle of quaternion algebras (cf. [P] §2).

The (generic fiber of the) conic bundle on X is determined by a symmetric 3×3 matrix with coefficients in the field of functions $k(X)$ of X. Diagonalizing this matrix we obtain a matrix $diag(f, g, h)$. The associated even Clifford algebra is the quaternion algebra with symbol $(-fg, -fh)$, i.e. it is the rank 4 algebra over $k(X)$ with generators $\mathbf{i}, \mathbf{j}$ and relations $\mathbf{i}^2 = -fg$, $\mathbf{j}^2 = -fh$, $\mathbf{ij} = -\mathbf{ji}$.

Conversely, to check that such an element of $Br(k(X))$ defines an element of $Br(X)$ one can use the sequence 6.7 or, equivalently, the tame symbol (cf. [Wi]). Wittenberg's explicit example [Wi] of elements of order two in the Brauer group of an elliptic fibration is related to a 'degenerate' case (the curve $\mathcal{C}$ has three irreducible components) of the construction considered in this section.

8. Remarks on general K3 surfaces.

8.1. **Two torsion in $Br(X)$.** Let X be a K3 surface with $NS(X) = \mathbf{Z}h$ and $h^2 = 2d > 0$. Then $Br(X)_2 \cong (\mathbf{Z}/2\mathbf{Z})^{21}$. As in the case of elliptic fibrations (cf. Proposition 3.3), there are various types of elements $\alpha \in Br(X)_2$ corresponding to the isomorphism classes of the sublattices of index two

$$\Gamma_\alpha = \ker(\alpha : T_X \longrightarrow \mathbf{Z}/2\mathbf{Z}), \qquad \alpha \in Br(X)_2,\ \alpha \neq 0.$$

To classify these sublattices we observe that the inclusion $NS(X) = \mathbf{Z}h \hookrightarrow \Lambda_{K3}$ is unique up to isometry. Hence we may assume:

$$NS(X) = \mathbf{Z}h \cong \mathbf{Z}(1, d) \hookrightarrow U \hookrightarrow U \oplus \Lambda' \cong H^2(X, \mathbf{Z}), \qquad \text{with} \quad \Lambda' = U^2 \oplus E_8(-1)^2.$$

Let $v=(1,-d)\in U$, so $v^2=-2d$. Then the transcendental lattice is:

$$T:=T_X\cong \mathbf{Z}v\oplus\Lambda'\cong\langle -2d\rangle\oplus U^2\oplus E_8(-1)^2.$$

Since Λ' is a unimodular (and hence self-dual) lattice, any $\alpha\in Br'(X)_2$ can be written as:

$$\alpha: T\longrightarrow \mathbf{Z}/2\mathbf{Z},\qquad nv+\lambda'\longmapsto a_\alpha n+\langle\lambda_\alpha,\lambda'\rangle \mod 2$$

for a unique $a_\alpha\in\{0,1\}$ and a $\lambda_\alpha\in\Lambda'$ whose class in $\Lambda'/2\Lambda'$ is uniquely determined by α.

To classify the index two sublattices of T_X we only need to determine their discriminant groups and discriminant forms ([N], 1.13.3).

8.2. **Proposition.** Let X be a K3 surface with $NS(X)=\mathbf{Z}h$ and $h^2=2d$. Let $\alpha\in Br'(X)_2=Hom(T,\mathbf{Z}/2\mathbf{Z})$, $\alpha\neq 0$, be defined by $a_\alpha\in\{0,1\}$ and $\lambda_\alpha\in\Lambda'$ as in 8.1. Let $\Gamma_\alpha=\ker(\alpha)$ be the index two sublattice of T defined by α. Then:

i) $\Gamma_\alpha^*/\Gamma_\alpha\cong\mathbf{Z}/2d\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$ if $a_\alpha=0$, there are $2^{20}-1$ such lattices.

ii) $\Gamma_\alpha^*/\Gamma_\alpha\cong\mathbf{Z}/8d\mathbf{Z}$ if $a_\alpha\neq 0$, there are 2^{20} such lattices.

   a) In case $d\equiv 0 \mod 2$, these 2^{20} lattices are isomorphic to each other.

   b) In case $d\equiv 1\mod 2$, there are two isomorphism classes of such lattices. One class, the even class, has $2^9(2^{10}+1)$ elements and is characterized by $\frac{1}{2}\langle\lambda_\alpha,\lambda_\alpha\rangle\equiv 0\mod 2$, the other (odd) class has $2^9(2^{10}-1)$ elements and is characterized by $\frac{1}{2}\langle\lambda_\alpha,\lambda_\alpha\rangle\equiv 1\mod 2$.

Proof. Since $\alpha\neq 0$, the discriminant of Γ_α is $2^2\mathrm{disc}(T)=8d$, and this is the order of $\Gamma_\alpha^*/\Gamma_\alpha$. Let $a=a_\alpha$.

In case $a=0$, we have an orthogonal direct sum decomposition $\Gamma_\alpha=\mathbf{Z}v\oplus\Gamma'_\alpha$ where $\Gamma'_\alpha=\ker(\alpha)\cap\Lambda'$. Hence $\Gamma_\alpha^*/\Gamma_\alpha$ is the direct sum of the discriminant group of $\mathbf{Z}v$ (which is $\mathbf{Z}/2d\mathbf{Z}$) and the discriminant group of Γ'_α, which has order 4. It is easy to see that $\lambda_\alpha/2\in\Gamma_\alpha^*/\Gamma_\alpha$ and that it has order two. Next choose a $\mu\in\Lambda'$ with $\langle\lambda_\alpha,\mu\rangle=1$, then also $\mu\in\Gamma_\alpha^*/\Gamma_\alpha$ has order two. This proves the case $a=0$.

In case $a=1$, the element $u:=(-v,2d\lambda_\alpha)/4d\in T\otimes\mathbf{Q}$ is in the lattice Γ_α^* since for $(nv,\lambda')\in\Gamma_\alpha$:

$$\langle u,(nv,\lambda')\rangle=\tfrac{1}{4d}(2dn+2d\langle\lambda_\alpha,\lambda'\rangle)=\tfrac{1}{2}(n+\langle\lambda_\alpha,\lambda'\rangle)=\tfrac{1}{2}\alpha(nv,\lambda')\in\mathbf{Z}.$$

Note that $4du=(-v,2d\lambda_\alpha)\not\in\Gamma_\alpha$ but $8du\in\Gamma_\alpha$. Hence u has order $8d$ in the discriminant group and thus it is a generator. The value of the discriminant form on this generator is:

$$q(u)=q\Big(\frac{(-v,2d\lambda_\alpha)}{4d}\Big)=\frac{-2d+4d^2\langle\lambda_\alpha,\lambda_\alpha\rangle}{16d^2}=\frac{-1+2d\langle\lambda_\alpha,\lambda_\alpha\rangle}{8d}\mod 2\mathbf{Z}.$$

To find the isomorphism classes, we need to see if there exists an integer x such that

$$x^2\cdot\frac{-1}{8d}\equiv\frac{-1+4d}{8d}\qquad\mod 2\mathbf{Z}.$$

Multiplying by $8d$ we get the equivalent condition:

$$x^2\equiv 1-4d\qquad\mod 16d.$$

If d is odd, this gives a contradiction when considered modulo 8. In case d is even, we write $d = 2^r e$ with e odd and $r \geq 1$. Then the equation becomes:

$$x^2 \equiv 1 - 2^{r+2}e \mod 2^{r+4}e.$$

Since e is odd, the equation is equivalent to $1 - 2^{r+2}e$ being a square modulo 2^{r+4} (which is true: any integer $\equiv 1 \bmod 8$ is a square modulo any power of two) and modulo e (which is trivial: $x^2 \equiv 1 - 2^{r+2}e \equiv 1 \bmod e$ is satisfied by $x = 1$). □

8.3. In contrast to the case when X has an elliptic fibration, not all of the sublattices Γ_α allow a primitive embedding into Λ_{K3}. In case Γ_α has a primitive embedding, this embedding is not necessarily unique ([O], [St]).

8.4. **Corollary.** The lattice Γ_α has a primitive embedding into Λ_{K3} if and only if $a_\alpha \neq 0$ and, in case $d \equiv 1 \bmod 2$, $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle \equiv 0 \bmod 2$.

Proof. If Γ_α has a primitive embedding i into Λ_{K3}, then we must have $i(\Gamma_\alpha)^\perp \cong \mathbf{Z}f$ for some f with $f^2 = 8d$ for reasons of rank and discriminant. Thus such an i exists if and only if the discriminant form of Γ_α is minus the one of $\mathbf{Z}f$. The computation done in the proof of Proposition 8.2 gives the result. □

8.5. **The case $h^2 = 2$.** If $h^2 = 2$, the map $\phi_h : X \to \mathbf{P}^2$ is a double cover of $\mathbf{P}^2$ branched over a smooth sextic curve $C \subset \mathbf{P}^2$, so the genus of C is ten. It was shown in [F], Theorem 2.4, that there is an exact sequence:

$$0 \longrightarrow Br(\mathbf{P}^2 - C)_2 \xrightarrow{\phi_h^*} Br(X) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

We recall the geometry involved and add some remarks on the subgroup $\phi_h^*(\mathbf{P}^2 - C)_2$.

8.6. **The case $h^2 = 2$ and α is even.** If $d = 1$, $a_\alpha \neq 0$ and $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle \equiv 0 \bmod 2$, the lattice Γ_α has a unique primitive embedding into Λ_{K3} and thus defines a unique K3 surface Y_α. It is easy to see that Y_α has a polarization of degree 8 and that Y_α has an embedding into $\mathbf{P}^5$ where it is the complete intersection of three quadrics (cf. [Mu2], [Kh]). The discriminant curve C of this net of quadrics is the plane sextic over which $X \to \mathbf{P}^2$ branches. The net of quadrics defines, and is defined by, an even theta characteristic on C (cf. [B2]). Thus one has a canonical bijection between the even theta characteristics on the genus ten curve C and the 'even' sublattices $\Gamma_\alpha \subset T_X$ (see also [L], [O'G] for this sublattice of index two). In [Bh] it is shown that Y_α is the moduli space of certain orthogonal bundles on X: this is a kind of inverse to Mukai's construction of X as a moduli space of bundles on Y_α.

8.7. **The case $h^2 = 2$ and α is odd.** If $d = 1$, $a_\alpha \neq 0$ and $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle \equiv 1 \bmod 2$, the lattice Γ_α does not have a primitive embedding into the K3 lattice. However, it is isometric to a primitive sublattice of $H^4(Z_\alpha, \mathbf{Z})$ where Z_α ($\subset \mathbf{P}^5$) is a cubic fourfold which contains a plane P. In fact,

$$\Gamma_\alpha(-1) \cong \langle h^2, P\rangle^\perp \qquad (\subset H^4(Z_\alpha, \mathbf{Z})),$$

where $h \in H^2(Z_\alpha, \mathbf{Z})$ is the hyperplane class, cf. [V], and the quadratic form on $\Gamma_\alpha(-1)$ is the opposite to the one on Γ_α. The projection of the fourfold Z_α from the plane P defines a quadric bundle structure on Z_α. The rulings of the quadrics define a double cover of $\mathbf{P}^2$, which is X. The quadric bundle defines, and is defined by, an odd theta characteristic on C (cf. [V], [L], [B2]) and in this way one obtains a canonical bijection between the even theta characteristics on the genus ten curve C and the 'odd' sublattices $\Gamma_\alpha \subset T_X$. For more interesting relations between K3 surfaces and cubic fourfolds see [H].

8.8. **The case $h^2 = 2$ and $a_\alpha = 0$.** In case $d = 1$ and $a_\alpha = 0$, we have $\alpha \in Br(\mathbf{P}^2 - C)_2$. The conic bundle P_α over X defined by α is obtained by pull-back from a conic bundle Q_α on $\mathbf{P}^2$ which ramifies along C. Thus Q_α defines an étale 2:1 cover of C, that is, an invertible sheaf $L_\alpha \in J(C)_2 - \{0\}$.

Conversely, the resolution of the sheaf $i_*L_\alpha(1)$ on $\mathbf{P}^2$, here $i : C \hookrightarrow \mathbf{P}^2$ is the inclusion, gives a symmetric 3×3 matrix M_α with entries which are homogeneous of degree two and whose determinant vanishes on C. Thus M_α defines the conic bundle Q_α (cf. [BCZ]). In this way one obtains canonical bijections between the points of order two in $J(C)$, the points of order two in $Br(\mathbf{P}^2 - C)$ and the sublattices $\Gamma_\alpha \subset T_X$ with $a_\alpha = 0$.

For $x, y \in \mathbf{C}^3$ we thus have a quadratic form ${}^t y M_\alpha(x) y$, which is a polynomial of bidegree (2,2) in x and y. Let $W_\alpha \subset \mathbf{P}^2 \times \mathbf{P}^2$ be the threefold defined by ${}^t y M_\alpha(x) y = 0$. The projections π_x and π_y define two conic bundle structures on W_α. In particular, we get two sextics, $C = C_x$ and C_y as branch curves and thus we get two K3 surfaces, branched over these curves, X and X_α.

On the other hand, the lattice $\Gamma_\alpha \cong < -2 > \oplus U(2) \oplus U \oplus E_8(-1)^2$ has exactly two embeddings as index two sublattice in the lattice T. In fact, these embeddings correspond to isotropic subgroups of the discriminant form of Γ_α. As Γ_α has discriminant form $(\mathbf{Z}/2\mathbf{Z})^3 \to \mathbf{Q}/2\mathbf{Z}$, $(\bar{x}, \bar{y}, \bar{z}) \mapsto \frac{1}{2}x^2 + yz \bmod 2\mathbf{Z}$, the isotropic subgroups are those generated by $(0, 1, 0)$ and $(0, 0, 1)$.

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Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italia
E-mail address: geemen@mat.unimi.it