The Schottky problem and second order theta functions.

Bert van Geemen

December 6, 1999

1 Introduction

The Schottky problem arose in the work of Riemann. To a Riemann surface of genus g one can associate a period matrix, which is an element of a space $\mathbf{H}_g$ of dimension $g(g+1)/2$. Since the Riemann surfaces themselves depend on only $3g-3$ parameters if $g \geq 2$, the question arises as to how one can characterize the set of period matrices of Riemann surfaces. This is the Schottky problem.

There have been many approaches, and a few of them have been succesfull. All of them exploit a complex variety (a 'ppav') and a subvariety, the theta divisor, which one can associate to a point in $\mathbf{H}_g$. When the point is the period matrix of a Riemann surface, this variety is known as the Jacobian of the Riemann surface. A careful study of the geometry and the functions on these varieties reveals that Jacobians and their theta divisors have various curious properties. Now one attempts to show that such a property characterizes Jacobians. We refer to [M2], Lectures III and IV for a nice exposition of four such methods, to [vdG], [B], [D2] for overviews of later results and [V] for a newer approach.

In these notes we discuss a particular approach to the Schottky problem which has its origin the work of Schottky and Jung (and unpublished work of Riemann). It uses the fact that to a genus g curve one can associate certain abelian varieties of dimension $g-1$, the Prym varieties. In our presentation we emphasize an intrinsic line bundle on a ppav (principally polarized abelian variety) and the action of a Heisenberg group on this bundle. A systematic study of the geometry associated to these leads in a natural way to Prym varieties. Moreover, one finds several other remarkable properties of Jacobians which suggest geometrical solutions of the Schottky problem.

Acknowledgements. I thank the organizers of the conference 'Variedas abelianas y funciones theta' for the oppertunity to present these lectures and for providing the pleasent working conditions in Morelia.

2 The Schottky problem

Introduction. We recall the basic results on period matrices of Riemann surfaces. References are [ACGH], [C], [GH] and [CGV]. Then we briefly discuss modular forms, a reference is [Ig1].

2.1 Period matrices

2.1.1 Let C be a Riemann surface of genus g (we consider only compact Riemann surfaces in these lectures). On the homology group $H_1(C,\mathbf{Z})\cong\mathbf{Z}^{2g}$ there is an (alternating, nondegenerate) intersection form. A symplectic basis of $H_1(C,\mathbf{Z})$ is a basis $\{\alpha_1,\ldots,\alpha_g,\ \beta_1,\ldots,\beta_g\}$ satisfying

$$(\alpha_i,\alpha_j)=(\beta_i,\beta_j)=0,\qquad(\alpha_i,\beta_j)=\delta_{ij},$$

with δ_{ij} Kronecker's delta (so $\delta_{ii}=1,\ \delta_{ij}=0$ if $i\neq j$).

The complex vector space of holomorphic one forms on C is denoted, as usual, by $H^0(C,\Omega_C)$, it has dimension g. Given a path γ in C and an $\omega\in H^0(C,\Omega_C)$ one can compute the integral $\int_\gamma\omega$. We will view $\int_\gamma$ as a map $H^0(C,\Omega_C)\to\mathbf{C}$, thus it is an element of $H^0(C,\Omega_C)^*$, the dual vector space of $H^0(C,\Omega_C)$. If γ is a closed path, the integral only depends on the homology class of γ, which we denote by the same symbol: $\gamma\in H_1(C,\mathbf{Z})$. Thus we get a map:

$$H_1(C,\mathbf{Z})\longrightarrow H^0(C,\Omega_C)^*,\qquad\gamma\longmapsto\int_\gamma.$$

This map is injective, in fact much more is true:

2.1.2 Theorem. Let $\{\alpha_i,\ \beta_j\}$ be a symplectic basis of $H_1(C,\mathbf{Z})$. Then there is a unique basis $\{\omega_1,\ldots,\omega_g\}$ of $H^0(C,\Omega_C)$ such that:

$$\int_{\alpha_i}\omega_j=\delta_{ij}.$$

Thus the elements $\int_{\alpha_i}\in H^0(C,\Omega_C)^*$ form the dual basis of the basis $\{\omega_j\}$ of $H^0(C,\Omega_C)$.

2.1.3 A symplectic basis of $H_1(C,\mathbf{Z})$ thus determines a basis of $H^1(C,\Omega_C)$. For this we only use the α_i. We now use the β_j to define a complex $g\times g$ matrix:

2.1.4 Definition. The period matrix of C (with respect to the symplectic basis $\alpha_i,\ \beta_j$ of $H_1(C,\mathbf{Z})$) is the matrix

$$\tau=(\tau_{ij})\in M_g(\mathbf{C})\quad\text{with}\quad\tau_{ij}:=\int_{\beta_i}\omega_j$$

and $\omega_j\in H^0(C,\Omega_C)$ as in Theorem 2.1.2.

2.1.5 Remark. The period matrix determines the image of $H_1(C,\mathbf{Z})$ in $H^0(C,\Omega_C)^*$. In fact, using the basis $\int_{\alpha_i}$ of $H^0(C,\Omega_C)^*$, we have

$$\int_{\beta_i}=\tau_{i1}\int_{\alpha_1}+\tau_{i2}\int_{\alpha_2}+\ldots+\tau_{ig}\int_{\alpha_g},$$

since both sides give the same result when applied to the basis elements $\omega_j\in H^0(C,\Omega_C)$.

2.1.6 Torelli's theorem. Torelli's theorem asserts that one can recover the Riemann surface from its period matrix. There are many proofs of this theorem, all of them use the Jacobian its theta divisor which are associated to a period matrix (see also 4.3.7).

2.2 The Siegel upperhalf space

2.2.1 The Schottky problem. The Schottky problem basically asks for equations which determine the period matrices of Riemann surfaces among all $g \times g$ matrices. There are two well known properties of period matrices: $\tau_{ij} = \tau_{ji}$ (so period matrices are symmetric) and $Im(\tau)$, the imaginary part of τ, which is a symmetric, real, $g \times g$ matrix, defines a positive definite quadratic form on $\mathbf{R}^g$: ${}^tx(Im\tau)x > 0$ for all $x \in \mathbf{R}^g$. We write $Im(\tau) > 0$.

This leads to the following definition and theorem.

2.2.2 Definition. The Siegel upperhalf space $\mathbf{H}_g$ is:

$$\mathbf{H}_g := \{\tau \in M_g(\mathbf{C}) : \ {}^t\tau = \tau, \quad Im(\tau) > 0 \ \}.$$

2.2.3 Theorem. Let τ be the period matrix of a Riemann surface. Then $\tau \in \mathbf{H}_g$.

2.2.4 The subset $\mathbf{H}_g$ of $M_g(\mathbf{C})$ is actually complex manifold (with the complex structure induced from that on $M_g(\mathbf{C})$). In fact, $\mathbf{H}_g$ is an open subset of the vector space of symmetric $g \times g$ matrices (if $Im(\tau) > 0$ then also $Im(\tau + \tau') > 0$ for any symmetric τ' with sufficiently small coefficients). The dimension of $\mathbf{H}_g$ is $\frac{1}{2}g(g+1)$.

We investigate $\mathbf{H}_g$ in more detail to see what kind of equations for period matrices one should expect.

2.2.5 The symplectic group. To define the period matrix of a Riemann surface, we had to choose a symplectic basis. Any two symplectic bases are related by an element of

$$\Gamma_g := Sp(2g, \mathbf{Z}) = \{A \in M_{2g}(\mathbf{Z}) : {}^tAE_0A = E_0 \ \} \quad \text{with} \quad E_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

One can show that Γ_g acts on $\mathbf{H}_g$ as follows:

$$\Gamma_g \times \mathbf{H}_g \longrightarrow \mathbf{H}_g, \qquad (A, \tau) \mapsto A\tau := (a\tau + b)(c\tau + d)^{-1}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, \ldots, d$ are $g \times g$ blocks of A. The period matrices of a Riemann surface X are a Γ_g-orbit in $\mathbf{H}_g$. Thus, rather than study the period matrices of Riemann surfaces in $\mathbf{H}_g$, one could study their images under the quotient map

$$\pi : \mathbf{H}_g \longrightarrow A_g := \Gamma_g \backslash \mathbf{H}_g.$$

The action of Γ_g on $\mathbf{H}_g$ is properly discontinuous (but not fixed points free) and A_g is complex variety (with singularieties if $g > 1$). If $g = 1$, $A_g \cong \mathbf{C}$ using the j-invariant for elliptic curves (=Riemann surfaces of genus 1).

2.2.6 Moduli spaces. Let M_g be the moduli space of Riemann surfaces of genus g. It is a variety whose points correspond to isomorphism classes of Riemann surfaces. Then we have a well defined holomorphic map:

$$j : M_g \longrightarrow A_g, \qquad [X] \longmapsto \Gamma_g \tau$$

where τ is a period matrix of X. Torelli's theorem implies that j is injective.

The Schottky problem can now be reformulated as the problem of finding equations for the image of j.

2.2.7 Definition. Let $J_g^0 \subset \mathbf{H}_g$ be the set of period matrices of Riemann surfaces. Its image in A_g is

$$j(M_g) = Image(J_g^0 \longrightarrow A_g = \Gamma_g \backslash \mathbf{H}_g).$$

The subvarieties J_g^0 and $j(M_g)$ are not closed, see 2.2.8. We define the Jacobi locus J_g to be the closure of J_g^0 in $\mathbf{H}_g$:

$$J_g := \overline{J_g^0} \qquad (\subset \mathbf{H}_g).$$

2.2.8 Decomposable matrices. A $\tau \in \mathbf{H}_g$ will be called decomposable if τ lies in the Γ_g orbit of matrices in diagonal block form (the diagonal blocks being matrices in upper half planes of lower dimension). The set $J_g - J_g^0$ in $\mathbf{H}_g$ consists of decomposable matrices, the diagonal blocks being period matrices of Riemann surfaces of lower genus. This follows from a result of Hoyt.

2.2.9 Modular forms. From Teichmüller theory one knows that the subset J_g is actually an irreducible subvariety of $\mathbf{H}_g$ of dimension $3g-3$ for $g > 1$, if $g = 1$ one has $\mathbf{H}_1 = J_1 = J_1^0$.

Since J_g is a complex subvariety of $\mathbf{H}_g$, it is natural to ask for holomorphic functions f_i on $\mathbf{H}_g$ such that $f_i(\tau) = 0$ for all i implies $\tau \in J_g$. The fact that J_g is invariant under Γ_g suggests that we could try to find such f_i which are Γ_g-invariant. It is known that a variant of this idea will work.

A Siegel modular form of weight k is a holomorphic function on $\mathbf{H}_g$ which transforms in the following way under Γ_g:

$$f : \mathbf{H}_g \longrightarrow \mathbf{C}, \qquad f(A\tau) = \det(c\tau + d)^k f(\tau), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g,$$

in case $g = 1$ one has to add a certain growth condition on $f(\tau)$ for $\tau \to i\infty$. The modular forms of weight k form a complex vector space which has finite dimension. For suitable, large, k, a basis $f_0, \ldots, f_N$ of this vector space gives an everywhere defined map:

$$\tilde{\mu}_k : \mathbf{H}_g \longrightarrow \mathbf{P}^N, \qquad \tau \longmapsto (f_0(\tau) : \ldots : f_N(\tau)).$$

Since $f_i(A\tau) = \det(c\tau + d)^k f_i(\tau)$ for each i, we have $\tilde{\mu}_k(A\tau) = \tilde{\mu}_k(\tau)$ and thus the map $\tilde{\mu}_k$ factors over $A_g = \Gamma_g \backslash \mathbf{H}_g$. In this way we get a map:

$$\mu_k : A_g = \Gamma_g \backslash \mathbf{H}_g \longrightarrow \mathbf{P}^N.$$

A fundamental result is that for suitable k the map μ_k embeds A_g (so $A_g \cong \mu_k(A_g)$). The image of μ_k (and thus A_g itself) is a quasi projective variety, that is, a Zariski open subset of a projective variety. For this projective variety one can take the Satake compactification of A_g, which has the following nice set-theoretic description as a disjoint union:

$$\overline{A_g} \cong A_g \,\dot{\cup}\, A_{g-1} \,\dot{\cup}\, \ldots \,\dot{\cup}\, A_0,$$

here A_0 is defined to be a point. Actually the maps μ_k extend to maps $\overline{A_g}$ and for suitable k one has $\overline{\mu_k(A_g)} \cong \overline{A_g} \cong \mu_k(\overline{A_g})$.

The closure of $\mu_k(j(M_g))$ is a projective subvariety of $\mathbf{P}^N$. Thus it is defined by a (finite) set of homogeneous polynomials. Determining these polynomials as well as the modular forms of weight k gives a solution to the Schottky problem. The 'best type' of equations for the Jacobi locus J_g are thus homogeneous polynomials in modular forms.

Note that if $F \in \mathbf{C}[X_0, \ldots, X_N]$ is a homogeneous polynomial of degree d and $f_0, \ldots, f_N$ is a basis of $M(\Gamma_g, k)$ then $\tau \mapsto F(f_0(\tau), \ldots, f_N(\tau))$ is a modular form of weight kd. Thus the equations for J_g we look for will be modular forms.

2.2.10 Algebraic geometry. We just explained that the nicest equations for the period matrices are modular forms. These modular forms are obtained from homogeneous polynomials which define the (closure in $\mathbf{P}^N$ of) the image of the composition

$$M_g \xrightarrow{j} A_g = \Gamma_g \backslash \mathbf{H}_g \xrightarrow{\mu} \mathbf{P}^N.$$

The map μ is given by modular forms on $\mathbf{H}_g$. To find the image of point $[C]$ ($\in M_g$), one has determine a period matrix $\tau \in \mathbf{H}_g$ of C and then evaluate the modular forms at τ. In practice these two 'transcendental' steps cannot be made explicit, except for very special cases, for instance when the Riemann surface C has many automorphisms.

However, any Riemann surface is an algebraic curve and can thus be defined by a polynomial $F \in \mathbf{C}[X, Y]$ in two variables. Using the algebraic geometrical approach one finds that the coordinates of $\mu(j[C])$ are given by polynomials in the coefficients of F. For example, if the curve is hyperelliptic (and $F = -Y^2 + \prod(X - a_i)$) Tomae's formulas essentially compute these coordinates (cf. [M3]). In the classical literature one finds several other partial results for more general curves.

2.2.11 Table. We conclude this section with a table. It shows that the Schottky problem is trivial for $g \leq 3$ and that for $g = 4$ one equation might be suffcient.

g	$\dim A_g$	$\dim J_g$	$\mathrm{codim}_{A_g} J_g$
2	3	3	0
3	6	6	0
4	10	9	1
5	15	12	3
g	$\frac{1}{2}g(g+1)$	$3g-3$	$\frac{1}{2}(g-2)(g-3)$

3 Abelian Varieties

3.1 Complex tori and polarizations

3.1.1 We return to the problem of finding the modular forms which solve the Schottky problem. It turns out that 'nature' has already done most of the hard work. To explain this we introduce abelian varieties and show how they are related to $\mathbf{H}_g$ and its quotient $A_g = \Gamma_g \backslash \mathbf{H}_g$.

In a sense, we take the longest possible route. However, it is an interesting one where we see various important geometrical objects and constructions. At the end we get, for free, an explicit map

$$\Theta : A_g(2,4) := \Gamma_g(2,4) \backslash \mathbf{H}_g \longrightarrow \mathbf{P}^{2^g-1}$$

which we will introduce and study in detail in the next chapter since it appears to be of great importance for the Schottky problem. The variety $A_g(2,4)$ is a finite covering of A_g.

The intrinsic approach which we sketch here is due to Mumford, but is implicit in the classical literature on theta functions. Readers already familiar with theta functions will find the classical results in the corresponding notation in the next chapter (and might want to skip this chapter).

We study line bundles on complex tori and we recall that A_g is the moduli space of principally polarized abelian varieties. Then we introduce the Heisenberg group and give some applications.

3.1.2 Complex tori. As a first step we will associate a geometric object, a complex torus to a $\tau \in \mathbf{H}_g$. Then we characterize the tori obtained in this way in Theorem 3.1.4.

Since $Im(\tau) > 0$, the matrix $Im(\tau)$ is invertible. The image of $\mathbf{Z}^{2g}$ in $\mathbf{C}^g$ under the $g \times 2g$ matrix $(I\,\tau)$ is then a lattice Λ_τ in $\mathbf{C}^g$:

$$\Lambda_\tau := \mathbf{Z}^g + \tau\mathbf{Z}^g \hookrightarrow \mathbf{C}^g.$$

The quotient of $\mathbf{C}^g$ by this lattice is denoted by

$$X_\tau := \mathbf{C}^g/\Lambda_\tau,$$

it is a (compact) complex variety, a torus. In case $\tau \in J_g^0$ is the period matrix of a Riemann surface C this torus is $J(C)$, the Jacobian of C. Using notation from the previous lecture we have:

$$J(C) := H^0(C, \Omega_C)^*/H_1(C, \mathbf{Z}).$$

3.1.3 The polarization. Let V be a complex g-dimensional vector space and let $X = V/\Lambda$ be a complex torus. Then we can choose a basis $B = \{a_1, \ldots, a_g,\, b_1, \ldots, b_g\}$ for the lattice Λ in such a way that $\{a_1, \ldots, a_g\}$ is a $\mathbf{C}$-basis of V. The matrix $\tau_B = \tau_{X,B} \in M_g(\mathbf{C})$ defined by $b_i = \sum_j (\tau_B)_{ij} a_j$ then has an invertible imaginary part, but $Im(\tau_B)$ need not be positive definite nor is τ_B necessarily symmetric.

We should recall however that the period matrix of a Riemann surface was defined with respect to a symplectic basis (w.r.t. the intersection form). The formulation of the following theorem is not so elegant since various conventions are not compatible. Note that if B is a symplectic basis for an alternating form E on Λ and we define

$$B' := \{a_1, \ldots, a_g, -b_1, \ldots, -b_g\} \qquad (\text{with} \quad B = \{a_1, \ldots, a_g,\, b_1, \ldots, b_g\})$$

then B' is a symplectic basis for the form $-E$.

3.1.4 Theorem. A complex torus $X = V/\Lambda$ has a basis B for which $\tau_B \in \mathbf{H}_g$ iff there is an alternating form $E : \Lambda \times \Lambda \to \mathbf{Z}$ such that

1. B is symplectic basis for $-E$,
2. The $\mathbf{R}$-linear extension of E to $\mathbf{C}^g = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ satisfies the two conditions:

   $$E(iv, iw) = E(v, w), \qquad E(v, iv) > 0 \qquad (v,\, w \in \mathbf{C}^g)$$

   and $v \neq 0$ in the second condition.

In case $\tau \in \mathbf{H}_g$, the basis $B = \{e_1, \ldots, e_g, \tau e_1 \ldots, \tau e_g\}$ of Λ_τ (with e_i the i-th standard basis vector of $\mathbf{C}^g$) and the alternating form E defined by the matrix $-E_0$ satisfy these two conditions.

Proof. Let B and E satisfy the conditions of the theorem and let B' be as above. Then B' is a symplectic basis of Λ and E is given by the matrix E_0 on the basis B'. Let $[i] : V \to V,\ x \mapsto ix$ be multipication by i ($i^2 = -1$) and let J_τ be the $2g \times 2g$ matrix of $[i]$ w.r.t. the $\mathbf{R}$-basis B' of V. The conditions in 2 translate in the following matrix identities:

$${}^tJ_\tau E_0 J_\tau = E_0, \qquad E_0 J_\tau > 0 \qquad \text{with} \quad E_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To determine J_τ we define $W = \mathbf{R}a_1 + \ldots + \mathbf{R}a_g$ ($\cong \mathbf{R}^g$) and

$$R : W \times W \longrightarrow V, \qquad (u, v) \longmapsto u + iv.$$

Since $[i](u + vi) = -v + ui$ we have $[i]R(u, v) = RJ(u, v)$ where $J : W^2 \to W^2$ is given by the matrix:

$$J := R^{-1}[i]R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By definition of τ_B, the map

$$\Omega_\tau := (I \ - \tau_B) : W^2 \longrightarrow V$$

maps (u, v) to $u_1 a_1 + \ldots u_g a_g + v_1(-b_1) + \ldots + v_g(-b_g)$. Thus

$$J_\tau = \Omega_\tau^{-1}[i]\Omega_\tau = (\Omega_\tau^{-1}R)(R^{-1}[i]R)(R^{-1}\Omega_\tau).$$

Let $\tau = X + iY$ and $X,\, Y \in M_g(\mathbf{R})$, then $\Omega_\tau(v_1, v_2) = v_1 - \tau v_2 = (v_1 - Xv_2) + i(-Yv_2) = R(v_1 - Xv_2, -Yv_2)$, so we have

$$R^{-1}\Omega_\tau = \begin{pmatrix} 1 & -X \\ 0 & -Y \end{pmatrix}, \qquad (R^{-1}\Omega_\tau)^{-1} = \begin{pmatrix} 1 & -XY^{-1} \\ 0 & -Y^{-1} \end{pmatrix} \qquad J_\tau = \begin{pmatrix} -XY^{-1} & Y + XY^{-1}X \\ -Y^{-1} & Y^{-1}X \end{pmatrix}.$$

The matrix ${}^tJ_\tau E_0 J_\tau$ is then:

$$\begin{pmatrix} {}^tY^{-1}({}^tX-X)Y^{-1} & {}^tY^{-1}Y+{}^tY^{-1}(X-{}^tX)Y^{-1}X \\ -{}^tYY^{-1}+{}^tX{}^tY^{-1}(X-{}^tX)Y^{-1} & -{}^tX{}^tY^{-1}Y+{}^tYY^{-1}X+{}^tX{}^tY^{-1}({}^tX-X)Y^{-1}X \end{pmatrix}.$$

The condition ${}^tJ_\tau E_0 J_\tau = E_0$ is equivalent to ${}^tX = X,\quad {}^tY = Y$ (since Y is invertible, $X = {}^tX$ iff we have a zero in the upper left corner, then we have a I upper right iff ${}^tYY^{-1} = I$ (i.e. $Y = {}^tY$), the rest follows).

The condition $EJ_\tau > 0$ is:

$$\begin{pmatrix} Y^{-1} & -Y^{-1}X \\ -XY^{-1} & Y+XY^{-1}X \end{pmatrix} > 0 \quad \text{equivalently,} \quad \begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix}\begin{pmatrix} Y^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix}\begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} > 0.$$

Since a matrix A is positive definite iff ${}^tSAS$ is positive definite (where S is invertible) iff ${}^tA^{-1} = {}^t(A^{-1})AA^{-1}$ is positive definite, this condition is equivalent to $Y > 0$. □

3.2 Line bundles.

3.2.1 To understand the significance of an alternating form with properties as in Theorem 3.1.4 we recall the basic facts on line bundles. Line bundles on a complex variety X are important in the study of maps from X to a projective space $\mathbf{P}^N$. There is a natural line bundle, usually denoted by $\mathcal{O}(1)$ on $\mathbf{P}^N$. Given a holomorphic map $\phi : X \to \mathbf{P}^N$ the pull-back of $\mathcal{O}(1)$ to X is a line bundle $L := \phi^*\mathcal{O}(1)$ on X. Moreover, the map ϕ is given by global sections of L. Thus a knowledge of line bundles and their sections allows one to determine all embeddings (if any) of X in a projective space. We restrict ourselves to a discussion of line bundles on a torus. References for line bundles on tori are [M1], Chapter 1 and [LB], Chapter 2.

3.2.2 Cocycles. To construct a map

$$\theta : X = V/\Lambda \longrightarrow \mathbf{P}^N$$

one could use holomorphic functions $f_i : V \longrightarrow \mathbf{C}$ $(0 \le i \le N)$ satisfying the transformation rules:

$$f_i(z+\lambda) = c_\lambda(z) f_i(z), \qquad (\lambda \in \Lambda,\ z \in V)$$

with c_λ independent of i and $c_\lambda(z) \neq 0$ for all $z \in V$. Then $(\ldots : f_i(z+\lambda) : \ldots) = (\ldots : f_i(z) : \ldots)$ and the map $V \to \mathbf{P}^N$, $z \mapsto (\ldots : f_i(z) : \ldots)$ factors over $X = V/\Lambda$. The transformation law implies that

$$c_{\lambda+\mu}(z) = c_\lambda(z+\mu)c_\mu(z)$$

(replace z by $z+\mu$ in the transformation rule). This is the cocycle rule and $\{c_\lambda\}_{\lambda\in\Lambda}$ is called a cocycle for Λ.

3.2.3 Line bundles. Given a cocycle $\{c_\lambda\}$ one can construct a complex variety L of dimension $1+\dim X$ as follows. Let Λ act on $V \times \mathbf{C}$ by the rule:

$$\lambda \cdot (z,t) := (z+\lambda, c_\lambda(z)t),$$

that $(\lambda+\mu)\cdot(z,t) = \mu\cdot(\lambda\cdot(z,t))$ follows from the cocycle condition. The orbit space $L := (V\times\mathbf{C})/\Lambda$ is a complex manifold and has a holomorphic map $\pi_L : L \to X$ induced by $(z,t)\mapsto z$. The triple (L,π,X) is a line bundle. Any line bundle on a torus can be defined by a cocycle.

3.2.4 Global sections and theta functions. A holomorphic function $f : V \to \mathbf{C}$ satisfying the rule $f(z+\lambda) = c_\lambda(z)f(z)$ defines a map

$$\tilde{s} : V \longrightarrow V\times\mathbf{C}, \quad z\mapsto (z,f(z)) \qquad \text{and} \quad \lambda\cdot\tilde{s}(z) = \tilde{s}(z+\lambda),$$

hence $\tilde{s}$ gives a well-defined holomorphic map $s : X \to L$ which obviously satisfies $(\pi_L s)(z) = z$ for all $z\in V$. Such maps $s : X\to L$ are called (holomorphic) sections of the line bundle L and such functions are called theta functions. Any section is obtained from a theta function. The global sections form a finite dimensional $\mathbf{C}$-vector space denoted by $\Gamma(X,L)$.

3.3 Classification of Line bundles

3.3.1 Isomorphism of bundles. Given a torus $X = V/\Lambda$ we now want to determine all line bundles on X as well as the vector spaces $\Gamma(X,L)$.

Two line bundles L, L' on X are isomorphic if there is a bi-holomorphic, fibre preserving map

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ \pi_L\downarrow & & \downarrow\pi_{L'} \\ X & \xrightarrow{id_X} & X \end{array} \qquad \pi_{L'}\phi = \pi_L$$

which is linear on the fibers. Let c_λ, c'_λ be the cocycles defining L and L'. Then ϕ corresponds to a bi-holomorphic map

$$\Phi : V\times\mathbf{C}\longrightarrow V\times\mathbf{C}, \qquad (z,t)\longmapsto (z+\lambda, \tilde{\phi}(z)t)$$

which intertwines the actions of Λ given by the two cocycles, so:

$$(z+\lambda,\, c_\lambda(z)\tilde{\phi}(z)t) = (z+\lambda,\, \tilde{\phi}(z+\lambda)c'_\lambda(z)t)$$

for all z, t, λ.

3.3.2 The Picard group. The set of line bundles on the torus X modulo isomorphism is an abelian group with tensor product as group law, equivalently, product $\{z\mapsto c_\lambda(z)c'_\lambda(z)\}_\lambda$ of cocycles. This group is called the Picard group of X, $Pic(X)$. It can be identified with the sheaf cohomology group $H^1(X,\mathcal{O}_X^*)$. From the exponential sequence:

$$0\longrightarrow\mathbf{Z}\longrightarrow\mathcal{O}_X\xrightarrow{e}\mathcal{O}_X^*\longrightarrow 0, \qquad e(f) := e^{2\pi i f}$$

one obtains an exact sequence:

$$0 \longrightarrow Pic^0(X) \longrightarrow Pic(X) \stackrel{c}{\longrightarrow} NS(X) \longrightarrow 0,$$

here $NS(X)$ is the Néron-Severi group of X (which is the subgroup $H^2(X,\mathbf{Z}) \cap H^{1,1}(X)$ of $H^2(X,\mathbf{Z}) \cong \mathbf{Z}^{g(2g-1)}$) and $Pic^0(X)$ ($\cong H^1(X,\mathcal{O})/H^1(X,\mathbf{Z})$) is a complex torus. The map c is the first Chern class of a line bundle. For $X = V/\Lambda$ the group $NS(X)$ is canonically isomorphic to the group of $\mathbf{Z}$-valued alternating bilinear forms E on Λ with $E(ix,iy) = E(x,y)$:

$$NS(X) = \{E \in Hom(\Lambda \times \Lambda, \mathbf{Z}), \quad E(x,y) = -E(y,x), \quad E(ix,iy) = E(x,y)\ \}.$$

The group $Pic^0(X)$ is canonically isomorphic to:

$$Pic^0(X) = Hom(\Lambda, U(1)), \qquad \text{with} \quad U(1) = \{z \in \mathbf{C} :\ |z| = 1\,\},$$

we do not discuss the complex structure on $Pic^0(X)$.

Given $\beta \in Hom(\Lambda, U(1))$ the corresponding line bundle L_β is defined by the cocycle $\{c_\lambda(z) := \beta(\lambda)\}_\lambda$, thus the cocycle does not depend on z. Actually any homomorphism $\gamma : \Lambda \longrightarrow \mathbf{C}^*$ defines a cocycle in this way, but the bundle L_γ is isomorphic to a unique L_β.

3.3.3 Appell-Humbert data. Given a $\beta \in Hom(\Lambda, U(1))$ and an $E \in NS(X)$ it is not possible in general to define canonically a line bundle on X. However, one can write down cocycles which exactly parametrize $Pic(X)$. These cocycles are determined by Appell-Humbert data which are defined as follows. For $E \in NS(X)$ define a (Hermitian) form

$$H = H_E : V \times V \longrightarrow \mathbf{C}, \qquad H(v,w) := E(v,iw) + iE(v,w)$$

(that H is Hermitian follows from $E(ix,iy) = E(x,y) = -E(y,x)$). Next one considers maps α (not homomorphisms in general):

$$\alpha : \Lambda \longrightarrow U(1), \qquad \alpha(\lambda + \mu) = \alpha(\lambda)\alpha(\mu)(-1)^{E(\lambda,\mu)}.$$

Note that if α, α' are such maps then $\alpha'\alpha^{-1}$ is a homomorphism $\Lambda \to U(1)$. A pair (α, H) is called Appell-Humbert data (for the torus X) and it defines a cocycle by:

$$c_\lambda(z) := \alpha(\lambda) e^{\frac{\pi}{2}H(\lambda,\lambda) + \pi H(\lambda,z)}.$$

Let $L_{(\alpha,H)}$ be the line bundle on X defined by this cocycle.

3.3.4 Theorem. (Appell-Humbert) Let X be a complex torus. Then each line bundle on X is isomorphic to a $L_{(\alpha,H)}$ for uniquely determined Appell-Humbert data (α, H).

3.3.5 Very ample line bundles It is now an interesting problem to determine the vector spaces $\Gamma(X, L_{(\alpha,H)})$ and to see for which bundles the global sections define an embedding of X in a projective space. Such line bundles are called very ample. The precise results are not so easy to state but the following result, due to Lefschetz, is classical. We recall that an Hermitian form H on a complex vector space V is called positive definite (and one writes $H > 0$) if $H(v,v) > 0$ for all $v \in V - \{0\}$. In terms of E $(= Im(H))$, the condition $H > 0$ is obviously:

$$E(v, iv) > 0 \qquad \forall v \in V - \{0\}.$$

3.3.6 Theorem. (Lefschetz) Let (α, H) be Appell-Humbert data on a torus X.

If $H > 0$ then the line bundle $L_{(\alpha^n, nH)}$ is very ample for any $n \geq 3$ (the bundle $L_{(\alpha,H)}$ is then called ample). Conversely, if $L_{(\alpha,H)}$ is very ample then $H > 0$.

3.3.7 Definition. A complex torus is called an abelian variety if it has a very ample line bundle, equivalently, if it has an embedding $\theta : X \to \mathbf{P}^N$. (In that case, Chow's theorem implies that the image $\theta(X)$ is a projective variety, that is, is defined by homogeneous polynomials.)

3.3.8 Riemann-Roch. The reader now recognises the two conditions on the polarization we met in 3.1.4. The first garantees that $E \in NS(X)$, the second that there exist very ample line bundles on X.

One can determine the dimension of $\Gamma(X, L_{(\alpha,H)})$ in terms of α and H. To state a weak version of this result, we recall that if $E : \Lambda \times \Lambda \to \mathbf{Z}$ is an alternating form then there is a (generalized symplectic) basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of Λ such that

$$E(a_j, a_k) = 0 = E(b_j, b_k), \qquad E(a_j, b_k) = e_j \delta_{jk},$$

with $\delta_{jk} = 0$ if $j \neq k$, $\delta_{jj} = 1$, and $e_j \in \mathbf{Z}_{\geq 0}$ with e_j dividing e_{j+1} (we adopt the convention that any e_i divides 0). These e_j are uniquely determined by E and are called the elementary divisors of E. In case $E(v, iv) > 0$ for all $v \neq 0$ we have of course $e_j \neq 0$ for all j.

3.3.9 Theorem. Let $L_{(\alpha,H)}$ be an ample line bundle on X (so $H > 0$) and let $e_1, \ldots, e_g$ be the elementary divisors of $E = Im(H)$. Then

$$\dim \Gamma(X, L_{(\alpha,H)}) = e_1 e_2 \ldots e_g.$$

3.4 Principally polarized abelian varieties.

3.4.1 The tori which interest us particularly are the Jacobians of curves and more generally, those defined by a $\tau \in \mathbf{H}_g$. Theorem 3.1.4 and the general results above show that such a torus $X = X_\tau$ comes with a given element $E \in NS(X)$ which defines ample line bundles $L_{(\alpha,H)}$ (with $H = H_E$) and E has elementary divisors $e_1 = \ldots = e_g = 1$ (so $\dim \Gamma(X, L_{(\alpha,H)}) = 1$). Note that $H > 0$ implies all $e_j \neq 0$ and thus: $\dim \Gamma(X, L_{(\alpha,H)}) = 1$ iff $e_j = 1$ for all j.

3.4.2 Translates of ample bundles For any $a \in X$ we have an isomorphism

$$T_a : X \longrightarrow X, \qquad x \longmapsto x + a,$$

the translation by a. One can pull-back line bundles along a translation. This gives a map $T_a^* : Pic(X) \to Pic(X)$ which preserves the Chern class $c(T_a^* L) = c(L)$. The map

$$\phi_L : X \longrightarrow Pic^0(X) = Hom(\Lambda, U(1)), \qquad a \longmapsto T_a^* L \otimes L^{-1}$$

is a homomorphism which is determined by $E = c(L)$ in the following way:

$$\phi_L(a) = [\lambda \longmapsto e^{2\pi i E(\tilde{a}, \lambda)}], \qquad (\lambda \in \Lambda)$$

where $\tilde{a} \in V$ maps to $a \in X = V/\Lambda$ (since $E(\Lambda, \Lambda) \subset \mathbf{Z}$ this does not depend on the choice of $\tilde{a}$). In case $det(E) \neq 0$, this map is surjective with kernel

$$\ker(\phi_L) = (\mathbf{Z}/e_1\mathbf{Z})^2 \oplus \ldots \oplus (\mathbf{Z}/e_g\mathbf{Z})^2.$$

In particular, if two ample line bundles L, L' have the same Chern class ($c(L) = c(L')$) then $L' \otimes L^{-1} \in Pic^0(X) = \phi_L(X)$ is isomorphic to $T_a^* L \otimes L^{-1}$ for some $a \in X$ and thus L and L' are translates of each other: $L' \cong T_a^* L$.

If L is ample, $det(E) \neq 0$ so the map ϕ_L induces an isomorphism on the tangent spaces at the origins. Since $Pic^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbf{Z})$ its tangent space at 0 is $H^1(X, \mathcal{O}_X)$ and we get $T_0 X \cong H^1(X, \mathcal{O}_X)$.

3.4.3 Definition. A principally polarized abelian variety (ppav for short) is a pair (X, E) with X a complex torus and $E \in NS(X)$ satisfying

$$H_E > 0, \qquad \dim \Gamma(X, L_{(\alpha, H_E)}) = 1.$$

Equivalently, a ppav is a pair (X, L) with X a complex torus and L an ample line bundle with $\dim \Gamma(X, L) = 1$, but ppavs (X, L), (X, L') will be identified if $c(L) = c(L')$, that is, if L and L' are translates. The dimension of a ppav (X, E) is defined to be $\dim X$.

Two ppav's (X, E), (X', E') are isomorphic, we write $(X, E) \cong (X', E')$, if there is an isomorphism $\phi : X \to X'$ with $\phi^* E' = E$, where $\phi^* : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is the map induced by ϕ.

3.4.4 Moduli of ppav's. With these definitions one can verify that $A_g = \Gamma_g \backslash \mathbf{H}_g$ is the moduli space of g-dimensional ppav's, roughly speaking:

$$A_g = \{(X, E)\}/\cong .$$

3.4.5 The theta divisor of a ppav. Given a ppav (X, E) of dimension g, any two line bundles L, L' with $c(L) = c(L') = E$ are translates and they have, upto scalar multiple, a unique non-zero global section. Thus the zero locus of such a section is a variety $\Theta = \Theta_{(X,E)}$ of dimension $g-1$ which depends only on (X, E) and is called the theta divisor of (X, E). In case $(X, E) = JC$ (with E the intersection form), Riemann proved that the divisor Θ_{JC} is isomorphic to the image of an Abel-Jacobi map:

$$\Theta_{JC} = Image(C^{(g-1)} \longrightarrow JC).$$

This divisor thus carries interesting geometrical information on the Riemann surface.

It is known that the dimension of the singular locus of Θ_{JC} is at least $g-4$. This property has been used to study the Schottky problem.

3.4.6 Symmetric line bundles. Given a ppav $(X = V/\Lambda, E)$ there is no canonical way to find a line bundle L on X with $c(L) = E$. However, one can consider Appell-Humbert data (α, H_E) with $\alpha : \Lambda \longrightarrow \{\pm 1\}$ $(\subset U(1))$. The corresponding bundles have the property that they are symmetric: $[-1]^* L \cong L$ where for $n \in \mathbf{Z}$:

$$[n] : X \longrightarrow X, \qquad x \longmapsto nx.$$

There are 2^{2g} such α and thus 2^{2g} such bundles, in fact if α defines a symmetric bundle, then so does $\xi\alpha$ where $\xi : \Lambda \to \{\pm 1\}$ is a homomorphism.

To see that such α indeed exist, let $B := \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a symplectic basis of Λ. We define

$$\alpha_B : \Lambda \cong \mathbf{Z}^g \times \mathbf{Z}^g \longrightarrow U(1), \qquad \alpha_B((m,n)) := (-1)^{{}^t m \cdot n}$$

where $(m, n) \in \mathbf{Z}^g \times \mathbf{Z}^g$ corresponds to $m_1 a_1 + \ldots + m_g a_g + n_1 b_1 + \ldots + n_g b_g \in \Lambda$ and ${}^t m \cdot n = m_1 n_1 + \ldots + m_g n_g$. It is easy to check that (α_B, H_E) are Appell-Humbert data and thus define a symmetric line bundle M_B on X and (X_τ, M_B) is a ppav.

3.4.7 An intrinsic line bundle. Although it is impossible to define intrisically a line bundle L on a ppav (X, E) with $c(L) = E$, one can define such a line bundle with $c(L) = 2E$. In fact, since $(-1)^{2E(\lambda,\mu)} = +1$, we have the line bundle

$$L := L_{(\alpha,H)}, \qquad Im(H) = 2E, \quad \alpha(\lambda) = 1 \quad (\forall \lambda \in \Lambda)$$

on X. This bundle is isomorphic to $M^{\otimes 2}$ for any symmetric line bundle with Chern class $c(M) = E$. Since the elementary divisors of $2E$ are $2e_i = 2$, we get:

$$\dim H^0(X, L) = 2^g.$$

This bundle will be very important in the remainder of these notes.

3.5 Heisenberg groups

3.5.1 Introduction. We now recall an interesting aspect of line bundles on an abelian variety. It permits one to find an intrinsic basis of the vector spaces $H^0(X, M)$ for any line bundle M on X. More precisely, one finds a finite set of such basis, and each such basis is defined up to multiplication by a constant. For the sake of simplicity we restrict ourselves to the case of the intrinsic bundle L on a ppav. References for this section are [Ig1], [LB], Chapter 6 and [K].

3.5.2 Let (X, E) be a ppav and let L be the intrinsic line bundle on X. Since $c(L) = 2E$ we get from 3.4.2 that

$$T_a^*L \cong L \iff \phi_L(a) = 0 \iff a \in X[2] := \ker([2] : X \longrightarrow X).$$

Given $a \in X[2]$ there is no intrinsic isomorphism $T_a^*L \to L$. If L is given by a cocycle $\{c_\lambda(z)\}$, then T_a^*L is given by the cocycle $\{c_\lambda(z + a)\}$ but since $c_\lambda(z) \neq c_\lambda(z + a)$, one has to choose a map $\tilde{\phi}$ as in 3.3.1 to get an isomorphism. This forces us to consider the Heisenberg group of L. Its elements are couples of a 2-torsion point and an isomorphism of bundles:

$$H(L) := \{(\phi, a) : \ a \in X[2],\ \phi : T_a^*L \xrightarrow{\cong} L\}.$$

With the natural group law, the Heisenberg group turns out to be non-abelian (!), in 3.5.3 we give a concrete description of $H(L)$. There is an exact sequence:

$$1 \longrightarrow \mathbf{C}^* \longrightarrow H(L) \longrightarrow X[2] \longrightarrow 0.$$

The first non trivial map is $t \mapsto (t, 0)$, the second is $(\phi, a) \mapsto \phi$. In fact, the map $H(L) \to X[2]$ is obviously surjective and the only isomorphisms of (any) line bundle on a compact complex variety with itself are scalar multiples of the identity. The subgroup $\mathbf{C}^*$ is the center of $H(L)$.

The group $H(L)$ acts on the vector space of global sections $\Gamma(X, L) \cong \mathbf{C}^{2^g}$ as follows:

$$(\phi, a)s := \phi(T_a^*s).$$

A basic fact is that this action is irreducible (the only invariant subspaces are $\{0\}$ and the space itself).

3.5.3 A concrete description of the Heisenberg group $H(L)$ and its action on $H^0(X, L)$ is obtained as follows. We define a group

$$H = H_g := \mathbf{C}^* \times (\mathbf{Z}/2\mathbf{Z})^g \times Hom_{\mathbf{Z}}((\mathbf{Z}/2\mathbf{Z})^g, \mathbf{C}^*)$$

with product (the term $m(u)$ makes it non-abelian):

$$(t, u, l)(s, v, m) = (tsm(u), u + v, l + m).$$

Note that the image of $\bar{1} \in \mathbf{Z}/2\mathbf{Z}$ by a homomorphism f to $\mathbf{C}^*$ must be 2-torsion, so $f(\bar{1}) = (-1)^n$ for a unique $n \in \mathbf{Z}/2\mathbf{Z}$. The map $f \mapsto n$ gives an isomorphism $Hom(\mathbf{Z}/2\mathbf{Z}, \mathbf{C}^*) \to \mathbf{Z}/2\mathbf{Z}$.

The Heisenberg group $H(L)$ is isomorphic to the group H. An isomorphism which is the identity on the subgroups $\mathbf{C}^*$ is called a theta structure:

$$\alpha : H(L) \stackrel{\cong}{\longrightarrow} H, \qquad \alpha_{|\mathbf{C}^*} = id_{\mathbf{C}^*}.$$

There are only a finite number of theta structures: the elements $(1, u, l) \in H$ have order at most 4 and, together with $\mathbf{C}^*$, generate H, moreover there are only a finite number of elements of order at most 4 in H (and thus in $H(L)$) so a theta structure is determined by the map it induces from the finite set of order at most 4 in $H(L)$ to the corresponding finite set in H.

The group H has a natural representation (the Schrödinger representation) on the vector space $V := Functions((\mathbf{Z}/2\mathbf{Z})^g, \mathbf{C})$ $(\cong \mathbf{C}^{2^g})$ as follows:

$$((t, u, l)f)(u') = tl(u')f(u + u') \qquad (t, u, l) \in H, \quad u' \in (\mathbf{Z}/2\mathbf{Z})^g.$$

The vector space V has a natural basis of 'delta functions' $\{\delta_\sigma\}_{\sigma \in (\mathbf{Z}/2\mathbf{Z})^g}$ with

$$\delta_\sigma : (\mathbf{Z}/2\mathbf{Z})^g \longrightarrow \mathbf{C}, \qquad u \longmapsto \begin{cases} 0 & \text{if} \quad u \neq \sigma \\ 1 & \text{if} \quad u = \sigma. \end{cases}$$

The main result on Heisenberg groups asserts that this representation of H on V coincides with the action of $H(L)$ on $H^0(X, L)$. The irreducibility of the representations and Schur's lemma imply that, given a theta structure $\alpha : H(L) \to H$, there is an essentially unique isomorphism $H^0(X, L) \cong V$ which intertwines the representations. The basis of delta functions in V then gives a canonical basis of the vector space $H^0(X, L)$. The consequences of this remarkable fact are discussed in the next chapter. We summarize the results in the theorem below.

3.5.4 Theorem. Given a theta structure $\alpha : H(L) \to H$, there is a unique (up to scalar multiple) isomorphism:

$$T(\alpha) : H^0(X, L) \longrightarrow V$$

which satisfies:

$$T(\alpha)\Big((\phi, a)s\Big) = \alpha(\phi, a)\,(T(\alpha)s) \qquad\qquad \begin{array}{ccc} H^0(X, L) & \stackrel{T(\alpha)}{\longrightarrow} & \mathbf{P}V \\ (\phi, a)\downarrow & & \downarrow\alpha(\phi, a) \\ H^0(X, L) & \stackrel{T(\alpha)}{\longrightarrow} & \mathbf{P}V \end{array}$$

In particular, the elements $T(\alpha)^{-1}(\delta_\sigma)$ are a basis of $H^0(X, L)$ and in these sections give a map

$$\Theta_\alpha : X \longrightarrow \mathbf{P}^{2^g - 1} = \mathbf{P}V,$$

where the coordinates of $\mathbf{P}V$ are parametrized by $(\mathbf{Z}/2\mathbf{Z})^g$.

3.5.5 We need one more fact. Since $(t, 0, 0) \in H$ acts by scalar multiplication on V, it acts trivially on $\mathbf{P}V$ and thus the Schrödinger representation induces a representation of $H/\mathbf{C}^* \cong (\mathbf{Z}/2\mathbf{Z})^{2g}$ on $\mathbf{P}V$. Let $\tilde{a} = (t, u, l) \in H$ map to $a \in H/\mathbf{C}^*$, then we write

$$U(a) : \mathbf{P}V \longrightarrow \mathbf{P}V$$

for the projective linear map induced by the action of $\tilde{a}$ on V.

For $x \in X[2]$ any two elements (ϕ, x), $(\phi', x) \in H(L)$ are related by $\phi' = t\phi$ for some $t \in \mathbf{C}^*$ and thus a theta structure induces an isomorphism (a level two structure), which we also denote by α (with some abuse of notation):

$$\alpha : X[2] \cong H(L)/\mathbf{C}^* \stackrel{\cong}{\longrightarrow} (\mathbf{Z}/2\mathbf{Z})^{2g} = H/\mathbf{C}^*.$$

With this notation, the following diagram is commutative:

$$\begin{array}{ccc} X & \stackrel{\Theta_\alpha}{\longrightarrow} & \mathbf{P}V \\ T_{\alpha^{-1}(a)}\downarrow & & \downarrow U(a) \\ X & \stackrel{\Theta_\alpha}{\longrightarrow} & \mathbf{P}V. \end{array}$$

so the translation by two-torsion points on X is given by projective transformations on $\mathbf{P}V$.

4 Geometry of second order theta functions

4.0.6 Introduction. We work out the consequences of the theory of the previous chapter and show that we recover some classical results from theta function theory. In particular, we find intrinsically defined maps $\Theta_\tau : X_\tau \to \mathbf{P}V$ and $\Theta : \mathbf{H}_g \to \mathbf{P}V$.

In section 4.2 we consider the intersection of the images of these maps. Somewhat surprisingly we find that if X_τ is a Jacobian the intersection is rather large. This is the first hint that second order theta functions are rather efficient at detecting Jacobians.

Next we consider the intersection of a tangent space of $\Theta(\mathbf{H}_g)$ with $\Theta(X_\tau)$, again the Jacobians behave in a peculiar manner. This led Izadi to a geometrical solution of the Schottky problem in the case $g = 4$.

4.1 Classical theta functions

4.1.1 Classical notation. We return to the upper half plane and the compex tori $X_\tau := \mathbf{C}^g/\Lambda_\tau$.

The standard basis of $\mathbf{Z}^{2g}$ gives a symplectic basis B of $\Lambda_\tau = \mathbf{Z} + \tau\mathbf{Z}^g$, the corresponding symmetric line bundle M_B will be denoted by M_τ (cf. 3.4.6). Thus (X_τ, M_τ) is a ppav. This bundle can thus be defined by a cocycle $\{c_\lambda\}$ as in 3.3.3 but Riemann's original cocycle $\{e_\lambda\}$ is more convenient for various purposes (for example, Riemann's cocycle is holomorphic as function of τ but the c_λ are not, since H is not holomorphic in τ). The explicit formula for Riemann's cocycle is:

$$e_\lambda(\tau, z) := e^{-\pi i({}^t k\tau k + 2{}^t kz)}, \qquad (\lambda = l + \tau k \ \in \Lambda_\tau).$$

The global section of M_τ is called Riemann's theta function. In general, the global sections of a line bundle M on X_τ with Chern class $c(M) = kc(M_\tau)$ for $k \in \mathbf{Z}_{>0}$ are called theta functions of level k.

The intrinsic bundle L (c.f. 3.4.7) over X_τ will be denoted by L_τ. Its defining cocycle is $\{e_\lambda^2\}$ (since $\alpha_B^2 = 1$).

4.1.2 Canonical basis. For applications of Theorem 3.5.4, which gives bases of $\Gamma(X_\tau, L_\tau)$, it is important to know that given $\tau \in \mathbf{H}_g$, the intrinsic line bundle L_τ on X_τ has a natural theta structure

$$\alpha_\tau : H(L_\tau) \longrightarrow H.$$

It has the property: $\alpha_\tau : (\phi, (a + \tau b)/2) \longmapsto (t_\phi, \bar{a}, \bar{b})$ for some $t_\phi \in \mathbf{C}^*$ and $\bar{b} \in (\mathbf{Z}/2\mathbf{Z})^g \cong Hom((\mathbf{Z}/2\mathbf{Z})^g, \mathbf{C}^*)$. The corresponding canonical basis of $H^0(X_\tau, L_\tau)$ is given by the second order theta functions:

$$\theta_\sigma(\tau, z) = \sum_{n \in \mathbf{Z}^g} e^{2\pi i({}^t(n+\frac{\sigma}{2})\tau(n+\frac{\sigma}{2}) + 2{}^t(n+\frac{\sigma}{2})z)}$$

where $\sigma \in (\mathbf{Z}/2\mathbf{Z})^g$ (and one may take representatives σ with components $\sigma_i \in \{0, 1\}$). That is, the isomorphism $T(\alpha_\tau)$ defined by the theta structure α_τ satisfies:

$$T(\alpha_\tau) : H^0(X_\tau, L_\tau) \xrightarrow{\cong} V, \qquad \theta_\sigma \longmapsto \delta_\sigma.$$

4.1.3 Maps. As a consequence of the previous results we now have, for any $\tau \in \mathbf{H}_g$, the natural map:

$$\Theta_\tau := \Theta_{\alpha_\tau} : X_\tau \longrightarrow \mathbf{P}V, \qquad z \longmapsto (\ldots : \theta_\sigma(\tau, z) : \ldots).$$

We see that the maps Θ_τ for various τ glue together, in fact the theta functions θ_σ are holomorphic in both z and τ. Thus we get a map from $\mathbf{H}_g \times \mathbf{C}^g$ to the projective space $\mathbf{P}V$. These theta functions and these maps were well known classically, but the approach with the Heisenberg group emphasizes that the construction is a canonical one.

4.1.4 Theta constants. Since an abelian variety $X = V/\Lambda$ has a 'canonical' point, the origin, any theta structure α defines a canonical point $\Theta_\alpha(0) \in \mathbf{P}V$. Thus we get a map Θ from the moduli space of pairs $((X, E), \alpha)$ of ppav's with theta structure to $\mathbf{P}V$, defined by: $\Theta((X, E), \alpha) := \Theta_\alpha(0)$.

In the classical picture, this gives the map

$$\Theta : \mathbf{H}_g \longrightarrow \mathbf{P}V, \qquad \tau \longmapsto \Theta_\tau(0) = (\ldots : \theta_\sigma(\tau, 0) : \ldots).$$

the coordinates are called theta constants (but they are not constant in τ (!)). Since there is more than one theta structure, one should not expect that this map factors over $A_g = \Gamma_g \backslash \mathbf{H}_g$. However this is 'almost' true.

For an even, positive, integer k we define (normal) subgroups (of finite index) of $\Gamma_g = Sp(2g, \mathbf{Z})$ by:

$$\Gamma_g(k) := \{A \in \Gamma_g : \; A \equiv I \bmod k\},$$

$$\Gamma_g(k,2k) := \left\{ A \in \Gamma_g(k) :\ A = I + k \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \text{ and } \mathrm{diag}(b') \equiv \mathrm{diag}(c') \equiv 0 \bmod 2k \right\}.$$

We denote by $A(H)$ the subgroup of automorfisms of H which are the identity on $\mathbf{C}^* \subset H$. If $\alpha,\ \alpha' : H(L) \to H$ are theta structures, the map $\alpha'\alpha^{-1} : H \to H$ is in $A(H)$ and for $\phi \in A(H)$ the map $\phi\alpha : H(L) \to H$ is a theta structure. In this way the set of theta structures is a principal homogeneous space under $A(H)$. There is an isomorphism of finite groups:

$$A(H) \cong \Gamma_g/\Gamma_g(2,4),$$

and the space $A_g(2,4) := \Gamma_g(2,4)\backslash\mathbf{H}_g$ is the moduli space of ppav's with a theta structure. The group $A(H)$ is the Galois group of the covering $A_g(2,4) \to A_g$.

4.1.5 Theorem. (Igusa) The map Θ factors over the quasi-projective variety $A_g(2,4) := \Gamma_g(2,4)\backslash\mathbf{H}_g$. The induced map, denoted by the same symbol,

$$\Theta : A_g(2,4) \longrightarrow \mathbf{P}V$$

has degree 1 on its image and its differential is injective at any point of $A_g(2,4)$. (It is not known if Θ is an embedding.) The closure $\overline{\Theta(\mathbf{H}_g)}$ of the image of Θ is a projective variety of dimension $g(g+1)/2$ in $\mathbf{P}V$.

4.1.6 Modular forms. Thus we have a very explicit map Θ of a finite cover of A_g to a projective space. The coordinate functions $\theta_\sigma(\tau,0)$ of the map Θ are modular forms of 'weight 1/2' (so basically transform with $\sqrt{det(a\tau+b)}$ but the sign of the root has to be specified). Taking the second Veronese image of this map (given by all products $\theta_\sigma\theta_\rho$) one gets a map whose coordinate functions are modular forms of weight 1. The theory of automorphisms of the Heisenberg group, or equivalently, the classical transformation laws for theta constants, imply that the finite group $\Gamma_g/\Gamma_g(2,4)$ acts on $\mathbf{P}V$ in such a way that Θ is an equivariant map:

$$\Theta(A\tau) = A\cdot\Theta(\tau) \qquad (A \in \Gamma_g).$$

Thus one can obtain modular forms for Γ_g by taking all homogeneous polynomials on $\mathbf{P}V$ which are invariant under the action $\Gamma_g/\Gamma_g(2,4)$ and substituting the θ_σ for the variables.

4.1.7 Kummer varieties. We consider now the map $\Theta_\tau : X_\tau \to \mathbf{P}V$. According to Lefschetz' Theorem, for $n \geq 3$ the map given by global sections of the bundle $M_\tau^{\otimes 3}$ on the the ppav (X_τ, M_τ) embed X_τ in a projective space. We are interested in the bundle $L_\tau = M_\tau^{\otimes 2}$ however.

The map Θ_τ given by the global sections of L is not injective. In case τ is indecomposable, the variety $\Theta(X_\tau)$ is isomorphic to the quotient of X_τ by the involution $x \mapsto -x$ (second order theta functions are even: $\theta_\sigma(\tau,-z)=\theta_\sigma(\tau,z)$, as one verifies from the Fourier series in 4.1.1). This quotient variety is called the Kummer variety of X_τ and will be denoted by

$$K_\tau := X_\tau/\pm 1.$$

The Kummer variety has 2^{2g} singular points, corresponding to the two-torsion points $X_\tau[2]$ (these are the fixed points of the involution).

$$\Theta_\tau(X_\tau)\cong K_\tau := X_\tau/\{\pm 1\}, \qquad Sing(K_\tau)=X_\tau[2].$$

In case τ is decomposable, the ppav X_τ is isomorphic to a product of lower dimensional ppav's and the $\Theta_{2,\tau}(X_\tau)$ is isomorphic to the product of their Kummer varieties ([K], Chapter 10).

4.2 Intersections

4.2.1 We face the obvious question: what is the intersection

$$\overline{\Theta(\mathbf{H}_g)}\cap\Theta_\tau(X_\tau) \qquad (\subset \mathbf{P}V)$$

between the moduli space and the Kummer variety? This question was raised in [GG], in view of the dimensions of the spaces involved, one wouldn't expect any intersection at all for $g>3$. From the definition of Θ it is however obvious that at least $\Theta_\tau(0)$, the origin of K_τ, lies in the intersection.

We consider the pre-image of this intersection in X_τ and call it Y_τ, so

$$Y_\tau := \Theta_\tau^{-1}\left(\overline{\Theta(\mathbf{H}_g)}\cap\Theta_\tau(X_\tau)\right).$$

In case τ is the period matrix of a Riemann surface, a relation between theta functions discovered by Fay implies that Y_τ has dimension at least two!

4.2.2 Recall that $J(C)=H^0(C,\Omega_C)^*/H_1(C,\mathbf{Z})$. For any $p,\,q\in C$ we can define an element in this space, simply denoted by $p-q$, as follows. Take any path γ in C starting in q and ending in p, then we get a map $\int_\gamma : H^0(C,\Omega_C)\to\mathbf{C}$. If we choose another path, $\int_\gamma$ will change to $\int_\gamma+\int_\alpha$ where α is a closed path, so α gives an element of $H_1(C,\mathbf{Z})$. Therefore the class of $\int_\gamma$ in $J(C)$ depends only on p and q and this is the desired $p-q\in J(C)$.

For $g>1$ one then obtains a surface in $J(C)$:

$$C-C := \{p-q\in J(C):\; p,\,q\in C\}.$$

With this notation Fay's result is:

4.2.3 Theorem. Let τ be the period matrix of a Riemann surface C. Then:

$$\{x\in J(C):\; 4x\in\; C-C\}\;\hookrightarrow\; Y_\tau.$$

4.2.4 Remarks. In case $g = 3$ one has $Y_\tau = (1/4)(C - C)$. It would be interseting to know for which τ one has $\dim Y_\tau > 0$ but I don't know of any results beyond $g = 3$.

We will consider an 'infinitesimal' version of this condition in 4.3.5 where we replace the moduli space by its tangent space.

One should note that the points of order 4 of $J(C) = X_\tau$ are contained in Y_τ (since $0 = p - p \in C - C$). A further study of these points, which we carry out in section 5.5, does lead to rather explicit Siegel modular forms (related to the classical Schottky-Jung relations) which give non-trivial equations for the period matrices of Riemann surfaces.

4.2.5 Sketch of proof of Theorem 4.2.3 Let $x \in J(C)$ with $4x = p - q$, let $D := 2x$, then $2(D + q) \equiv p + q$ (linear equivalence of divisors). Interpreting $J(C)$ as $Pic^0(C)$, the variety of line bundles of degree 0 on C, this implies that $\mathcal{O}_C(D + q)^{\otimes 2} \cong \mathcal{O}(p + q)$. Let s be the global section of $\mathcal{O}(p + q)$ which is zero in p and q. Inside the global space of the line bundle $\mathcal{O}_C(D + q)$ we can now consider the subvariety C_D of points ξ satisfying $\xi \otimes \xi \in s$. The subvariety C_D is an irreducible curve (provided $p \neq q$) and the bundle projection induces a 2:1 map $\pi : C_D \to C$ which ramifies only over p and q. Thus the genus of C_D is $2g$. The pull-back map $\pi^* : H^0(C, \Omega_C) \to H^0(C_D, \Omega_{C_D})$ induces a 'Norm' map $Nm : J(C_D) \to J(C)$. The kernel of Nm is an abelian variety of dimension g and has a principal polarization induced by the one on $J(C_D)$. This ppav is called the Prym variety of the cover π.

Thus $\ker(Nm) \cong X_\pi$ for some $\pi \in \mathbf{H}_g$. Fay [Fa] proves that, for a suitable pair of $\tau,\ \pi \in \mathbf{H}_g$ with $J(C) \cong X_\tau$ and $\ker(Nm) \cong X_\pi$ one has:

$$\Theta_\tau(x) = \Theta_\pi(0) \qquad (= \Theta(\pi) \in \overline{\Theta(\mathbf{H}_g)}).$$

Therefore $x \in Y_\tau$. The case $p = q$ follows by taking the limit $p \to q$. In that case the cover will become singular (and in fact the Prym variety in that case is best seen as a ppav of dimension $g - 1$).

4.3 Local intersections

4.3.1 Tangent spaces. Let $\tau \in \mathbf{H}_g$ be indecomposable. Then the image of $\Theta_\tau : X_\tau \to \mathbf{P}V$ is isomorphic to the Kummer variety $K_\tau := X_\tau / \pm 1$. The point $\Theta_\tau(0) = (\ldots : \theta_\sigma(\tau, 0) : \ldots)$ is singular on K_τ. Let

$$t_0 := \Theta_\tau(0) = (\ldots : \theta_\sigma(\tau, 0) : \ldots), \qquad t_{kl} := (\ldots : \frac{\partial^2 \theta_\sigma}{\partial z_k \partial z_l}(\tau, 0) : \ldots)_{\sigma \in \{0,1\}^g} \qquad (\in \mathbf{P}V).$$

(The second order theta functions $z \mapsto \theta_\sigma(\tau, z)$ are all even, thus the first order deriviatives vanish.) The embedded tangent space of $\Theta_\tau(X_\tau)$ at $\Theta_\tau(0)$ is the span of these points:

$$\mathbf{T}_{K_\tau,0} := \langle t_0, \ldots, t_{kl}, \ldots \rangle_{1 \leq k \leq l \leq g}, \qquad \dim \mathbf{T}_{K_\tau,0} = \tfrac{1}{2} g(g + 1).$$

The classical Heat equations for second order theta functions:

$$\frac{\partial^2 \theta_\sigma}{\partial z_k \partial z_l}(\tau, z) = 4\pi i (1 + \delta_{kl}) \frac{\partial \theta_\sigma}{\partial \tau_{kl}}(\tau, z)$$

for all $\tau \in \mathbf{H}_g,\ z \in \mathbf{C}^g$ and $i^2=-1$, these are easy to verify from the series definition of the θ_σ in 4.1.2. Therefore we also have:

$$t_{kl} = 4\pi i(1+\delta_{kl})(\ldots : \frac{\partial\theta_\sigma}{\partial\tau_{kl}}(\tau,0) : \ldots)_\sigma.$$

The vectors on the righthand side of the equation above and t_0 together obviously span the embedded tangent space $\mathbf{T}_{\Theta(\mathbf{H}_g),\Theta(\tau)}$ to $\Theta(\mathbf{H}_g)$ at $t_0=\Theta(\tau)$. Thus this space coincides with $\mathbf{T}_{K_\tau,0}$, we denote it by $\mathbf{T}_\tau$:

$$\mathbf{T}_\tau := \mathbf{T}_{K_\tau,0} = \mathbf{T}_{\Theta(\mathbf{H}_g),\Theta(\tau)}.$$

4.3.2 Remark. This may be seen as a geometric version of an intrinsic isomorphism. The Kummer variety K_τ is singular, but its tangent space is still defined by $(\mathbf{m}/\mathbf{m}^2)^*$ where $\mathbf{m} \subset \mathcal{O}_{K_\tau,0}$ is the maximal ideal in the local ring at $0 \in K_\tau$. This local ring is the ring of invariants (under $z \mapsto -z$) of the local ring $\mathcal{O}_{X_\tau,0} \cong \mathcal{O}_{\mathbf{C}^g,0}$. Thus $\mathcal{O}_{K_\tau,0}$ may be identified with the ring of even, convergent power series in g variables. Any $f \in \mathcal{O}_{K_\tau,0}$ has a Taylor series expansion:

$$f(z) = f(0) + \sum_{ij} a_{ij}z_iz_j + H.O.T.$$

We obviously have:

$$f \in \mathbf{m} \iff f(0)=0, \qquad \text{and} \quad f \in \mathbf{m}^2 \iff (f(0)=0,\ a_{ij}=0\ \ \forall i,j).$$

Thus $(\mathbf{m}/\mathbf{m}^2)^*$ is spanned by the monials z_iz_j. Each z_i is an element of $T^*_{X_\tau,0}$, thus we find:

$$T_{K_\tau,0} \cong S^2T_{X_\tau,0} \cong S^2H^1(X_\tau,\mathcal{O}),$$

the last isomorphism comes from the principal polarization on X_τ as in 3.4.2.

On the other hand, the first order deformations of an algebraic variety X are parametrized by $H^1(X,T_X)$. Since the tangent bundle of X_τ is trivial, $T_{X_\tau} \cong T_{X_\tau,0}\otimes\mathcal{O}_{X_\tau}$ and thus $H^1(X_\tau,T_{X_\tau}) \cong H^1(X_\tau,\mathcal{O})\otimes H^1(X_\tau,\mathcal{O})$. The deformations of X_τ parametrized by $\mathbf{H}_g$ preserve the polarization. This gives an identification:

$$T_{\mathbf{H}_g),\tau} = S^2H^1(X_\tau,\mathcal{O}), \qquad \text{and so} \qquad T_{\mathbf{H}_g),\tau} \cong T_{K_\tau,0}.$$

4.3.3 Γ_{00}**.** The (linear) projective subvariety $\mathbf{T}_\tau \subset \mathbf{P}V$ of dimension $\frac{1}{2}g(g+1)$ is defined by $(2^g-1)-\frac{1}{2}g(g+1)$ linear equations. We identify these equations in the following way.

Let $H=\sum_\sigma a_\sigma X_\sigma$ be the equation of a hyperplane in $\mathbf{P}V$. Then we can pull-back H to X_τ along the map $\Theta_\tau : X_\tau \to \mathbf{P}V$. This pull-back is the theta function $\theta_H := \sum_\sigma a_\sigma\theta_\sigma(\tau,z)$:

$$\Theta_\tau^* : H^0(\mathbf{P}V,\mathcal{O}(1)) \longrightarrow H^0(X_\tau,L_\tau), \qquad H \longmapsto \theta_H.$$

Then we find that $t_0=\Theta_\tau(0)$ lies in H iff $\theta_H(0)=0$ and similarly, $t_{ij}\in H$ iff $\partial^2/\partial z_i\partial z_j\theta_H(0)=0$. Therefore the defining equations for $\mathbf{T}_\tau$ are the elements of the vector space:

$$\Gamma_{00} := \left\{\theta \in H^0(X_\tau,L_\tau) :\ m_0(\theta)\geq 4\ \right\},$$

where m_0 stands for multiplicty at zero (since these theta functions are even, the multplicity is also even). That is,

$$\mathbf{T}_\tau \subset H \iff \theta_H \in \Gamma_{00}.$$

4.3.4 Explicit equations. We will need the following observation later. Let $P \in \mathbf{C}[\ldots, X_\sigma, \ldots]$ be a homogeneous polynomial which is zero on the projective variety $\overline{\Theta(\mathbf{H}_g)}$ ($\subset \mathbf{P}V$). Then

$$H_P := \sum_\sigma \frac{\partial P}{\partial X_\sigma}(a) X_\sigma, \qquad \text{with} \quad a = \Theta(\tau) = (\ldots, \theta_\sigma(\tau, 0) : \ldots)$$

is a linear form on $\mathbf{P}V$ which is zero on the tangent space $\mathbf{T}_\tau$ at a to $\overline{\Theta(\mathbf{H}_g)}$. Thus its restriction to $\Theta(X_\tau)$ gives a section in Γ_{00}. In case a is a smooth point of $\overline{\Theta(\mathbf{H}_g)}$, these linear forms cut out $\mathbf{T}_\tau$ ($\subset \mathbf{P}V$) and the corresponding sections span Γ_{00}.

4.3.5 The local intersection. We will now investigate an infinitesimal version of the intersection $\Theta_\tau(X_\tau) \cap \overline{\Theta(\mathbf{H}_g)}$ (see 4.2.3). That is, we replace $\overline{\Theta(\mathbf{H}_g)}$ by $\mathbf{T}_\tau$, its embedded tangent space at $\Theta(\tau)$.

We define:

$$F_\tau := \{x \in X_\tau : \ \theta(x) = 0 \quad \forall \theta \in \Gamma_{00}\}.$$

Then by the previous discussion F_τ is the pre-image of this intersection:

$$\Theta_\tau(F_\tau) = \Theta_\tau(X_\tau) \cap (\cap_{\theta_H \in \Gamma_{00}} H) = \Theta_\tau(X_\tau) \cap \mathbf{T}_\tau.$$

For period matrices of Riemann surfaces this intersection is again large (and it is related in a somewhat surprising way with the intersection $\Theta_\tau(X_\tau) \cap \overline{\Theta(\mathbf{H}_g)}$).

The incusion $\subset$ in the following theorem was observed by Fay, Gunning, van Geemen and van der Geer, the inclusion $\supset$ is due to Welters.

4.3.6 Theorem. Let $X_\tau = J(C)$, the Jacobian of a Riemann surface C. Then:

$$F_\tau = C - C \qquad \text{except if}$$

we are in the case that $g = 4$ and that C has two distinct line bundles of degee 3 with $\dim H^0 = 2$ (this is the case for the generic curve of genus 4). In that case we write the points in $Pic^3(C)$ corresponding to these bundles by g^1_3 and h^1_3 and we have $F_\tau = C - C \ \cup \{\pm(g^1_3 - h^1_3)\}$.

4.3.7 The surface $C - C \subset J(C)$ is singular at the origin $0 \in J(C)$. Its tangent cone there (viewed as subvariety of $\mathbf{P}T_{J(C),0}$) is the canonically embedded embedded curve. Thus if C is non hyperelliptic, Theorem 4.3.6 gives a proof of Torelli's theorem (it can also be used to prove Torelli for HE curves with a little bit of extra work).

4.3.8 For dimension reasons, one does not expect that $\dim F_\tau > 0$ for a $\tau \in \mathbf{H}_g$. This leads to the following conjecture, which was proved in case $g = 4$ by Izadi [Iz]. For $g = 4$ we thus have a geometrical solution to the Schottky problem. More refined versions of the conjecture and variants are discussed in [GG], [BD] and [D1].

4.3.9 Conjecture. Let $\tau \in \mathbf{H}_g$ be indecomposable, then $\dim F_\tau > 0 \quad \stackrel{?}{\Longleftrightarrow} \quad \tau \in J_g$.

4.3.10 Example. The case $g=3$ is easy to understand. Fix an indecomposable $\tau \in \mathbf{H}_3$ and let $\mathbf{T}=\mathbf{T}_\tau$. Then $\dim \mathbf{T}=6$ so $\mathbf{T}$ is a hyperplane in $\mathbf{P}V$. The corresponding section $\theta_{\mathbf{T}}$ spans Γ_{00} and its zero locus must be $C-C$. One can also verify directly that $C-C\in|2\Theta|$, the linear system defined by $L_\tau^{\otimes 2}$. We have $m_0(\theta_{\mathbf{T}})=4$ and if $\theta_{\mathbf{T}}(z)=F_4(z)+H.O.T$ is the Taylor series of $\theta_{\mathbf{T}}$, then the quartic polynomial F_4 is the defining equation for the canonical curve C if $X_\tau=J(C)$ with C non hyperelliptic. In case C is hyperelliptic one has $F_4=F_2^2$ and $C-C$ is the surface $C^{(2)}-g_2^1$, counted with multiplicity two; its tangent cone is the rational normal curve of degree $g-1$ (counted with multiplicity two).

5 Schottky-Jung relations

Introduction. We consider the eigenspaces of the linear maps $U(a):\mathbf{P}V\to\mathbf{P}V$ for $a\in(\mathbf{Z}/2\mathbf{Z})^{2g}=H/\mathbf{C}^*$, and, of course(!), the intersection of these eigenspaces with both the moduli space $\overline{\Theta(\mathbf{H}_g)}$ (in 5.1.3) and with the Kummer varieties $\Theta_\tau(X_\tau)$ in $\mathbf{P}V$ (in 5.1.4).

This leads us to a geometrical picture of the classical Schottky-Jung relations. We show how these relations can be used to construct to modular forms which are zero on the locus of period matrices of Riemann surfaces. Next we recall the known results.

5.1 Eigenspaces of Heisenberg group elements

5.1.1 Let $a\in H/\mathbf{C}^*$ be a point of order two, let and let $U(a):\mathbf{P}V\to\mathbf{P}V$ be the projective transformation defined in 3.5.5. We denote by $a_\tau\in X_\tau[2]$ the point for which the following diagram commutes:

$$\begin{array}{ccc} X_\tau & \stackrel{\Theta_\tau}{\longrightarrow} & \mathbf{P}V \\ T_{a_\tau}\downarrow & & \downarrow U(a) \\ X_\tau & \stackrel{\Theta_\tau}{\longrightarrow} & \mathbf{P}V \end{array} \qquad a_\tau := \alpha_\tau^{-1}(a)$$

here α_τ is the canonical theta structure which induces an isomorphism

$$\alpha_\tau : X[2] \stackrel{\cong}{\longrightarrow} H/\mathbf{C}^*.$$

5.1.2 Going down. Since $U(a)^2=id_{\mathbf{P}V}$, any lift of $U(a)$ to a linear map $\tilde{U}(a):V\to V$, satisfies $\tilde{U}(a)^2=\lambda I$ for a non-zero $\lambda\in\mathbf{C}$. The map $\tilde{U}(a)$ gives the action of an element $(t,u,l)\in H$ on V. The explicit formula for the action of H on V (see 3.5.3), shows that if $a\neq 0$, the map $\tilde{U}(a)$ has two eigenspaces $V_a^\pm$ in V. Their projectivizations $\mathbf{P}V_a^+$ and $\mathbf{P}V_a^-$ will be called the eigenspaces of $U(a)$. The signs are not defined intrinsically, any element in an eigenspace of $U(a)$ is fixed by $U(a)$.

The eigenspaces of $U(a)$ have dimension $2^{g-1}-1$ (they are projectivizations of linear spaces of dimension $2^{g-1}=\frac{1}{2}2^g$). The action of $H=H_g$ on V induces an action of a similar group H_{g-1} on the spaces $V_a^\pm$. As in Theorem 3.5.4 and section 4.1.3 this allows us to define for $\pi\in\mathbf{H}_{g-1}$ and the corresponding theta structure $\alpha_\pi : H(L_\pi)\to H_{g-1}$ maps (and similar ones with a $-$ sign):

$$\Theta_\pi^{a+}:X_\pi\longrightarrow\mathbf{P}V_a^+,\qquad \Theta^{a+}:\mathbf{H}_{g-1}\longrightarrow\mathbf{P}V_a^+,\qquad \Theta^{a+}(\pi):=\Theta_\pi^{a+}(0).$$

5.1.3 Boundary components. The images of $\mathbf{H}_{g-1}$ in the eigenspaces nicely fit in the 'holes' of the image of $\mathbf{H}_g$ in $\mathbf{P}V$, in fact we have disjoint union:

$$\overline{\Theta(\mathbf{H}_g)} = \Theta(\mathbf{H}_g) \mathbin{\dot\cup} \left(\cup_{a\in H/\mathbf{C}^*-\{0\}} \overline{\Theta^{a+}(\mathbf{H}_{g-1})} \cup \overline{\Theta^{a-}(\mathbf{H}_{g-1})}\right)$$

(the various $\overline{\Theta^{a\pm}(\mathbf{H}_{g-1})}$ may still intersect). The $\overline{\Theta^{a\pm}(\mathbf{H}_{g-1})}$ are images the $2(2^{2g}-1)$ boundary components of the Satake compactification of $A_g(2,4)$. It is known that ([vG]):

$$\overline{\Theta^{a+}(\mathbf{H}_{g-1})} = \overline{\Theta(\mathbf{H}_g)} \cap \mathbf{P}V_a^+.$$

5.1.4 Intersection. Let again $a \in H/\mathbf{C}^* - \{0\}$ and the corresponding $a_\tau \in X_\tau[2]$. We consider

$$\Theta_\tau(X_\tau) \cap \mathbf{P}V_a^+ \qquad (\subset \mathbf{P}V),$$

the intersection of a Kummer variety with an eigenspace of $U(a)$. Since $p \in \mathbf{P}V_a^+ \cup \mathbf{P}V_a^-$ iff $U(a)(p) = p$, and since $U(a)\Theta_\tau(x) = \Theta_\tau(x + a_\tau)$ we get:

$$\Theta_\tau(x) \in \mathbf{P}V_a^+ \cup \mathbf{P}V_a^- \iff \Theta_\tau(x + a_\tau) = \Theta_\tau(x).$$

Now assume that τ is indecomposable, then $\Theta_\tau(X_\tau) \cong K_\tau = X_\tau/\pm 1$. Thus

$$\Theta_\tau(x) \in \mathbf{P}V_a^+ \cup \mathbf{P}V_a^- \iff (x + a_\tau = x \quad \text{or} \quad x + a_\tau = -x).$$

The first condition is impossible, but the second gives $2x = a_\tau$, so x is a point of order 4. The number of points x with $2x = a$ is 2^{2g} (the difference of any two such is a point of order two). These map to $\frac{1}{2}2^{2g} = 2^{2g-1}$ points in the Kummer variety K_τ and each eigenspace $\mathbf{P}V_a^\pm$ gets $\frac{1}{2}2^{2g-1} = 2^{2(g-1)}$ points.

We conclude that $\Theta_\tau(X_\tau) \cap \mathbf{P}V_a^+$ is a set of $2^{2(g-1)}$ points for indecomposable τ's. Each of these sets is an orbit of the group H_{g-1}. Since each eigenspace also contains a boundary component of $\overline{\Theta(\mathbf{H}_g)}$, we introduce the following definition:

5.1.5 Definition of Schottky-Jung relations. A pair (X_τ, x) with $x \in X_\tau[2]-\{0\}$ satisfies the Schottky-Jung relations if for some $y \in X_\tau$ with $2y = x$ we have:

$$\Theta_\tau(y) \in \overline{\Theta^{a+}(\mathbf{H}_{g-1})} \qquad (\subset \mathbf{P}V_a^+),$$

here $a = \alpha_\tau(x)$ and $\mathbf{P}V_a^+$ is the eigenspace which contains $\Theta_\tau(y)$.

This condition depends only on the point $x \in X = X_\tau$, not on the choice of τ or y. In fact $\mathbf{P}V_a^+$ is one of the two eigenspaces of translation by x on $\mathbf{P}V$ (cf. 3.5.5). The Heisenberg group permutes the y's while it stabilizes $\overline{\Theta^{a+}(\mathbf{H}_{g-1})}$ (or maps it to $\overline{\Theta^{a-}(\mathbf{H}_{g-1})}$).

5.2 Examples

5.2.1 We inspect the Schottky-Jung relations for low genus. In the cases $g = 1,\ 2,\ 3$ the answer is easy. In case $g = 4$ we find a more interesting situation however.

5.2.2 $g=1$. The abelian variety $X_\tau = \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ is an elliptic curve. The line bundle $L_\tau \cong \mathcal{O}(2O)$ with $O \in X_\tau$ the origin. The Schrödinger representation of $H(2) = \mathbf{C}^* \times (\mathbf{Z}/2)^2$ (as sets) on V is given by (see 3.5.3)

$$(t,1,0) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \qquad (t,0,1) \mapsto \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \qquad (t,1,1) \mapsto \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix},$$

the matrices are with respect to the basis of δ-functions. The eigenspaces $\mathbf{P}V^\pm$ of these elements are points in $\mathbf{P}V \cong \mathbf{P}^1$, they are respectively:

$$(1:1),\ (1:-1), \qquad (1:0),\ (0:1), \qquad (1:i),\ (1-i)$$

with $i^2=-1$.

If we denote the image of the origin by $(a:b)$:

$$\Theta_\tau : X_\tau \longrightarrow \mathbf{P}V, \qquad (a:b) := \Theta_\tau(0),$$

then, using the equivariance of Θ_τ for the action of $X_\tau[2]$ and $H/\mathbf{C}^*$ we find that

$$\Theta_\tau(\tfrac{1}{2}+\tau 0) = \Theta_\tau(0+(\tfrac{1}{2}+\tau 0)) = (t,1,0)\cdot\Theta_\tau(0) = (b:a),$$

and similarly the images of the other two-torsion points can be determined:

$$\Theta_\tau(0+\tau\tfrac{1}{2}) = (a:-b), \qquad \Theta_\tau(\tfrac{1}{2}+\tau\tfrac{1}{2}) = (b:-a).$$

Since $\Theta_\tau(X_\tau) \cong X_\tau/\pm 1$, the image of $X[2]$ consists of 4 distinct points. Therefore a and b are non-zero. Then we can write these image points as:

$$(\lambda:1), \qquad (\lambda^{-1}:1), \qquad (-\lambda,1), \qquad (-\lambda^{-1}:1) \qquad (\lambda = a/b).$$

The elliptic curve X_τ can be recovered as the 2:1 cover of $\mathbf{P}^1$ ramified in these 4 points, thus an affine equation for X_τ is:

$$X_\tau : \qquad y^2 = x^4 - (\lambda^2+\lambda^{-2})x^2 + 1.$$

Given any point $(\lambda:1) \in \mathbf{P}V$ which is not one of the six eigenspaces, we get in this way an elliptic curve which is isomorphic to an X_τ, and $\Theta_\tau(0) = (\lambda:1)$.

By definition, $\Theta(\tau) = \Theta_\tau(0)$ so we get:

$$\Theta(\mathbf{H}_1) = \mathbf{P}^1 - \{6 \text{ eigenspaces}\}.$$

In fact, in this case $A_1(2,4) \cong A_1(4) \cong \overline{\Theta(\mathbf{H}_1)}$. The 6 eigenspaces $\mathbf{P}V_a^\pm$ correspond to to the boundary components so, formally, each is a $\Theta^{a\pm}(\mathbf{H}_0)$.

We recall that we have shown that the inverse image under Θ_τ of a point $\mathbf{P}V_a^\pm \in \mathbf{P}V$ is a point of order four in X_τ for any $\tau \in \mathbf{H}_g$. Thus as τ moves in $\mathbf{H}_1$, the images of these points remain fixed! Since the image of any point of of order 4 is a boundary component, all pairs (X_τ, x) satisfy the Schottky-Jung relations.

5.2.3 $g = 2$. Let $\tau \in \mathbf{H}_2$ be indecomposable. Then Θ_τ maps X_τ onto its Kummer surface K_τ in $\mathbf{P}V$:

$$\Theta_\tau : X_\tau \longrightarrow K_\tau \subset \mathbf{P}V \cong \mathbf{P}^3.$$

The surface K_τ has degree 4 and its singular consists of 16 points, the images of the two-torsion points.

Each eigenspace $\mathbf{P}V_a^+ = \mathbf{P}^1$ is now a projective line, we get $2 \cdot 15 = 30$ such lines in $\mathbf{P}^3$. These lines can be found easily (as in the $g = 1$ case) and they form a quite interesting, symmetric, configuration. We just observe that each line is a copy of $\overline{\Theta(\mathbf{H}_1)}$ and thus has 6 marked points on it. Through each such point there pass 2 other lines (and the lines only intersect in such points).

Since the Kummer surface has degree 4, it will meet each eigenline in 4 points, which are the images of points of order 4. This shows that any pair (X_τ, x) with $\tau \in \mathbf{H}_2$ satisfies the Schottky-Jung relations.

In case τ is decomposable, $X_\tau \cong X_\pi \times X_{\pi'}$ for some $\pi,\ \pi' \in \mathbf{H}_1$. Then $\Theta_\tau(X_\tau) \cong K_\pi \times K_{\pi'} \cong \mathbf{P}^1 \times \mathbf{P}^1$, which is embedded as a smooth quadric in $\mathbf{P}^3$. There are 10 quadrics which arise as image of a decomposable X_τ (one can recover the pair $\pi,\ \pi'$ from the image of the origin of X_τ in $\mathbf{P}^1 \times \mathbf{P}^1$). Any two of these 10 quadrics intersect in 4 lines, these are eigenlines.

As in the $g = 1$ case, the image of $\mathbf{H}_2$ is the complement of the eigenspaces and is isomorphic to the moduli space:

$$A_g(2,4) \cong \Theta(\mathbf{H}_2) := \mathbf{P}V - \{30 \text{ eigenspaces}\}.$$

The space $\mathbf{P}V$ is isomorphic to the Satake compactification of $A_g(2,4)$. The image in $\mathbf{P}V$ of the locus of the period matrices of Riemann surfaces is

$$M_2(2,4) := \Theta(J_2^0) = \mathbf{P}V - \{10 \text{ quadrics}\}$$

(these 10 quadrics are the images of decomposable abelian varieties). It can be shown that the universal cover of $M_2(2,4)$ is the Teichmüller space T_2.

The finite group $G := \Gamma_2/\Gamma_2(2,4)$ is the symmetry of the configuration of 30 lines and 10 quadrics, it sits in an exact sequence:

$$0 \longrightarrow (\mathbf{Z}/2\mathbf{Z})^4 \longrightarrow G \longrightarrow S_6 \longrightarrow 0,$$

where S_6 is the symmetric group (the 6 is related to the 6 Weierstrass points on a genus two curve). The action of the subgroup $(\mathbf{Z}/2\mathbf{Z})^4$ on $\mathbf{P}V$ coincides with the action of $H(2)/\mathbf{C}^*$ on $\mathbf{P}V$. This subgroup fixes the 10 quadrics and fixes the pairs of eigenlines (but can permute $\mathbf{P}V_a^+$ and $\mathbf{P}V_a^-$). The group S_6 permutes the 15 pairs of lines like it permutes the 15 (unordered) subsets with two elements $\{i,\ j\} \subset \{1,\ldots,6\}$, it permutes the 10 quadrics like the 10 (unordered) pairs of complementary (unordered) subsets with three elements $\{\,\{i,j,k\},\ \{l,n,m\}\,\}$ with $\{i,j,k\} \cup \{l,n,m\} = \{1,\ldots,6\}$.

5.2.4 $g = 3$. In this case each eigenspace $\mathbf{P}V_a^+ \cong \mathbf{P}^3$ is a linear subspace of $\mathbf{P}V \cong \mathbf{P}^7$. We know that for indecomposable $\tau \in \mathbf{H}_3$, the threefold $\Theta_\tau(X_\tau)$ intersects $\mathbf{P}V_a^+$ in 16 points. From the $g = 2$ example we know that $\mathbf{P}V_a^+ = \overline{\Theta^{a+}(\mathbf{H}_2)}$. Since any point of order 4 maps to an eigenspace, its image lies in a boundary component $\overline{\Theta^{a+}(\mathbf{H}_2)}$. Thus again we get ('for free') that for all $\tau \in \mathbf{H}_3$ and $x \in X_\tau$ of order two, the pair (X_τ, x) satisfies the Schottky-Jung relations.

The 6-fold $\overline{\Theta(\mathbf{H}_3)}$ ($\subset \mathbf{P}^7$) has degree 16, its defining equation is given in [GG].

5.2.5 $g = 4$. Now a point y of order 4 maps to an eigenspace $\mathbf{P}V_a^+$ of dimension 7, and may or may not lie on the 6-fold $\overline{\Theta^{a+}(\mathbf{H}_3)}$. We will discuss this situation in the next lectures.

5.2.6 Remark. For $g \leq 3$ the indeomposable abelian varieties are all Jacobians. For a Jacobian X_τ (of any dimension g) we know from 4.2.3 that $\Theta(X_\tau) \cap \overline{\Theta(\mathbf{H}_g)}$ contains a surface and that this surface contains all the points of order 4 (these are $(\frac{1}{4})0$). Since τ is indecoposable, $\Theta_\tau(X_\tau) \cap \mathbf{P}V_a^+$ consists of images of points of order 4. As $\overline{\Theta(\mathbf{H}_g)} \cap \mathbf{P}V_a^+ = \overline{\Theta^{a+}(\mathbf{H}_{g-1})}$ it is clear that, for any choice of period matrix τ for the Jacobian and any $x \in X[2] - \{0\}$, the pair (X_τ, x) satisfies the Schottky-Jung relations. We sketch another proof in 5.3.3.

5.3 Schottky Loci

5.3.1 We define two loci (algebraic subsets of the quasi projective variety A_g) of importance for the study of the Schottky-Jung (SJ) relations. These were introduced by Donagi who showed that there is an interesting difference between them when $g = 5$. The first one is:

$$S_g^{big} = \{[\tau] \in A_g = \Gamma_g \backslash \mathbf{H}_g : \exists x \in X_\tau[2] - \{0\} \text{s.t. } (X_\tau, x) \text{ satisfies the SJ relations } \}.$$

For a given ppav X, we ask for at least one point of order two a such that (X, a) satisfies the Schottky-Jung relations. We can also ask for the ppav's X such that for all $a \in X[2] - \{0\}$ we have the Schottky-Jung relations:

$$S_g^{small} = \{[\tau] \in A_g = \Gamma_g \backslash \mathbf{H}_g : (X_\tau, x) \text{ satisfies the SJ relations } \forall x \in X_\tau[2] - \{0\} \}.$$

The Schottky locus S_g as defined in [vG] coincides with S_g^{small}.

We already observed the following result in 5.2.6, other proofs were given by Schottky and Jung, Rauch and Farkas, Fay, and Mumford.

5.3.2 Theorem. Let $\overline{j(M_g)} \subset A_g$ be the closure of the locus of Jacobians $j(M_g)$ in A_g. Then:

$$\overline{j(M_g)} \subset S_g^{small} \quad (\subset S_g^{big}).$$

5.3.3 Scketch of proof of Theorem 5.3.2. Since S_g^{ssmall} is closed, it suffices to show $j(M_g) \subset S_g^{ssmall}$. Let C be a Riemann surface of genus g, choose a $\tau \in \mathbf{H}_g$ with $X_\tau = Jac(C) = H^0(C,\Omega)^*/H_1(C,\mathbf{Z})$ and let $x \in X_\tau[2] - \{0\}$. We will construct a ppav P_x of dimension $g-1$ such that there exists a $\pi \in \mathbf{H}_{g-1}$ with $P_x \cong X_\pi$ and, with $a = \alpha_\tau(x)$,

$$\Theta^{a+}(\pi) \in \Theta_\tau(X_\tau) \cap \mathbf{P}V_a^+,$$

which implies the Schottky-Jung relations (if one of the 2^{2g-2} points on the right is in $\overline{\Theta^{a+}(\mathbf{H}_{g-1})}$ then all of them are). To construct P_x one constructs first an unramified double cover of the Riemann surface.

The point of order two $x \in J(C) \cong Pic^0(C)$ corresponds to a line bundle L on C with $L \otimes L \cong \mathcal{O}$. As in 4.2.5, we get a curve $C_x \subset L$, the 'square root of 1'. The bundle projection $L \to C$ restricts to an unramified 2:1 map $\pi_x : C_x \to C$. Thus the genus of C_x is $2g-1$. The Prym variety of the covering π_x is defined to be $\ker(Nm : J(C_x) \to J(C))^0$ with Nm the map induced by π_x and ^{0} stands for connected component of the origin (in this case $\ker(Nm)$ has two components).

Another way to construct the unramified cover $\pi_x : C_x \to C$ is to use the theory of covering spaces and fundamental groups. Note that

$$X_\tau[2] \cong \tfrac{1}{2}H_1(C,\mathbf{Z})/H_1(C,\mathbf{Z}) \cong H_1(C,\mathbf{Z})/2H_1(C,\mathbf{Z}).$$

Using the intersection form $(*.*)$ on $H_1(C,\mathbf{Z})$, the point x defines a codimension one subspace $x^\perp$ in this $\mathbf{F}_2$-vector space:

$$x^\perp := \{b \in X_\tau[2] : \ (x.b) = 0 \bmod 2\}.$$

The (surjective) Hurewich map $\pi_1(C) \longrightarrow \pi_1(C)^{ab} \cong H_1(C,\mathbf{Z})$ composed with the 'mod 2' map gives a surjection $\phi : \pi_1(C) \longrightarrow X_\tau[2]$. Then $G_x := \phi^{-1}(x^\perp)$ is a subgroup of index two of $\pi_1(C)$. The quotient of the universal cover of C by G_x is a 2:1 unramified cover of C and this is C_x (moreover, $G_x = \pi_1(C_x)$). Now P_x is defined as above.

Next one studies the theta functions on the Jacobian of C_x and restricts them to the image of $J(C)$ in $J(C_x)$ and to P_x. The main result is the Schottky-Jung proportionalities, which for suitable period matrices τ of $J(C)$ and π of P_x state that $\Theta_\tau(x/2) \in \mathbf{P}V$ coincides with $\Theta_\pi(0) \in \mathbf{P}V_a^+ \subset \mathbf{P}V$.

These proportionalities are derived in [C], 6.4, p. 173, with $x = \frac{1}{2}(1,0,\ldots,0) \in \subset \mathbf{C}^g$. Then $y = \frac{1}{2}x \in \mathbf{C}^g$ corresponds to a point in X_τ with $2y = x$ and thus maps to an eigenspace of $U(\alpha_\tau(x))$. Explicit computations and elementary manipulations show:

$$\Theta_\tau(y) = (\ldots : \theta_\sigma(\tau,y)\ldots) \qquad \text{with} \quad \left\{ \begin{array}{lcl} \theta_{(0,\sigma_2,\ldots,\sigma_g)}(\tau,y) & = & \theta[^{0,\sigma_2,\ldots,\sigma_g}_{1,0,\ldots,0}](0,2\tau) \\ \theta_{(1,\sigma_2,\ldots,\sigma_g)}(\tau,y) & = & 0 \end{array} \right.$$

were we mix our notation on the left with Clemens' notation on the right, and we took $p = q$ in Clemens' formula. That formula, the Schottky-Jung proportionalities, shows:

$$(\ldots : \theta[^{0,\sigma'}_{1,0}](0,2\tau) : \ldots) = (\ldots : \theta[^{\sigma'}_{0}](0,2\pi) : \ldots)$$

where the index σ' runs over $(\mathbf{Z}/2\mathbf{Z})^{g-1}$. In our notation, we have:

$$\theta_{\sigma'}(\pi,0)=\theta[{}^{\sigma'}_{0}](0,2\pi).$$

To get our geometrical interpretation of the Schottky-Jung proportionalities, note that the eigenspace $\mathbf{P}V_a^+$ which contains $\Theta_\tau(y)$ is defined by: $X_\sigma=0$ if $\sigma_1=1$. The natural coordinates on this eigenspace are the $X_{\bar\sigma}$ which are induced by the $X_{(0,\bar\sigma)}$ on $\mathbf{P}V$. Thus the natural map

$$\Theta^{a+}:\mathbf{H}_{g-1}\longrightarrow \mathbf{P}V_a^+\ (\subset \mathbf{P}V)$$

is given by

$$\pi\longmapsto(\ldots:c_\sigma(\pi):\ldots)_{\sigma\in(\mathbf{Z}/2\mathbf{Z})^g},\qquad \text{with}\quad c_\sigma(\pi)=\begin{cases}\theta_{(\sigma_2,\ldots,\sigma_g)}(\pi,0) & \text{if } \sigma_1=0\\ 0 & \text{if } \sigma_1=1.\end{cases}$$

Thus we see that $\Theta_\tau(y)=\Theta^{a+}(\pi)$ where $\pi\in\mathbf{H}_{g-1}$ is a period matrix of the Prym variety P_x. □

5.4 Results

5.4.1 Here are the three main results on the Schottky loci. The first shows that the Schottky problem is solved in genus 4. For essentially trivial reasons, for $g\geq 5$ the big Schottky locus is really bigger than the small Schottky locus, and thus cannot coincide with the Jacobi locus. The second, surprising, result is of Donagi who shows that there is in $g=5$ also an interesting difference between the big and small Schottky locus. Finally one can show that for any g the Schottky locus is 'locally' equal to $\overline{j(M_g)}$.

We observe that for $g>4$ it is *not* known if $\overline{j(M_g)}$ and S_g coincide. (For example, we do not even know if they coincide for $g=5$, Donagi's example shows that this may not be so easy to decide).

5.4.2 Theorem. (Schottky, Igusa, Freitag) We have:

$$S_4^{small}=S_4^{big}$$

and this locus is defined by one (explicitly known) modular form (of weight 8) for $\Gamma_4=Sp(8,\mathbf{Z})$. Moreover ([Ig2], [Fr]):

$$\overline{j(M_4)}=S_4^{small}.$$

5.4.3 Theorem. For $g\geq 5$ we have $S_g^{small}\neq S_g^{big}$.

Proof. Let $\tau\in\mathbf{H}_4-J_4$ and let $\pi\in J_{g-4}$. Then (the isomorphism class of) $X:=X_\tau\times X_\pi$ in A_g is not in S_g^{small} but it is in S_g^{big}. (For $a=(a',0)\in X[2]$ the pair (X,a) does not satisfy the Schottky-Jung relations, but any pair $(X,(0,a''))$ will satisfy the Schottky-Jung relations). □

5.4.4 Theorem. (Donagi [D1]) Let $Y \subset \mathbf{P}^4$ be a smooth cubic threefold. Let $J(Y) := H^{2,1}(Y)^*/H_3(Y,\mathbf{Z})$ be its intermediate Jacobian, $J(Y)$ is a ppav of dimension 5. Then $J(Y) \notin J_5$, $J(Y)$ is not decomposable (i.e. is not a product of lower dimensional ppav's) but $J(Y) \in S_5^{big}$.

Proof. The theta divisor of $J(Y)$ has just one singular point [B3]. Since the theta divisor of a Jacobian has a singular locus of dimension $\geq g-4=1$ and a decomposable ppav has $\dim Sing(\Theta) = g-2 = 3$, we get the first two statements. For the most interesting result, $[J(Y)] \in S_5^{big}$ one uses the following relation, discovered by Donagi.

Let C be Riemann surface and let $a,\, b \in J(C)[2]-0$ with $E(a,b)=0$ where $E: J(C)[2]\times J(C)[2] \rightarrow \mathbf{Z}/2\mathbf{Z}$ is the Weil pairing which induced by the polarization. Let $P_a \cong X_{\pi(a)}$, $P_b \cong X_{\pi(b)}$ be the corresponding Pryms. One can prove that $P_a[2] \cong a^\perp/ < a >$ with $a^\perp$ as in the proof of 5.3.2, and we have a similar result for b. Thus we get points $\bar{b} \in P_a[2]$ and $\bar{a} \in P_b[2]$. Donagi proves that one can choose points $u \in P_a$ with $2u = \bar{b}$ and $v \in P_b$ with $2v = \bar{a}$ such that:

$$\Theta^{\bar{b}+}_{\pi(a)}(u) = \Theta^{\bar{a}+}_{\pi(b)} \qquad (\in \mathbf{P}V_a^+ \cap \mathbf{P}V_b^+ \subset \mathbf{P}V)$$

(strictly speaking, we should write $\mathbf{P}V^+_{\alpha_{\pi(a)}}$ etc.)

The space $\mathbf{P}V_a^+ \cap \mathbf{P}V_b^+ \cong \mathbf{P}^{2^{g-2}-1}$ is an eigenspace $\bar{b} \in H_{g-1}$, the Heisenberg group acting on $\mathbf{P}V_a^+$ (and also for an $\bar{a}$ in the copy of H_{g-1} acting on $\mathbf{P}V_b^+$). Then we have the natural map

$$\Theta^{\bar{a},\bar{b},+} : \mathbf{H}_{g-2} \longrightarrow \mathbf{P}V_a^+ \cap \mathbf{P}V_b^+.$$

Donagi's relation implies: if $(P_a,\bar{b})$ satisfies the Schottky-Jung relations then also $(P_b,\bar{a})$ satisifes them (because $\Theta^{\bar{b}+}_{\pi(a)}(u) \in \overline{\Theta^{\bar{a},\bar{b},+}(\mathbf{H}_{g-2})}$ implies $\Theta^{\bar{a}+}_{\pi(b)} \in \overline{\Theta^{\bar{a},\bar{b},+}(\mathbf{H}_{g-2})}$).

Next Donagi chooses a genus 6 curve C (a plane quintic) and $a,\ b$ in such a way that $P_a = J(D)$ for some genus 5 curve D and P_b is the intermediate Jacobian of a cubic threefold. Thus $(P_a,\bar{b})$ obviously satisfies the Schottky-Jung relations and one concludes that so does $(P_b,\bar{a})$. □

5.4.5 The locus S_g^{small} is an algebraic subset of A_g, and thus it is a finite union of irreducible components. One would like to show that it has only one component and that this component is $\overline{j(M_g)}$. The next theorem shows that is the case. The proof which we sketch in 5.5.2 doesn't give much information on the existence of other components.

5.4.6 Theorem. (van Geemen) The variety $\overline{j(M_g)}$ is an irreducible component of S_g^{small}.

5.5 Equations for the Schottky loci

5.5.1 We show how one expresses the Schottky-Jung relations in terms of modular forms, these are used in the proof of 5.4.6.

For $a \in H/\mathbf{C}^*$ we can represent $a_\tau := \alpha_\tau^{-1}(a) \in X_\tau$ by $\frac{1}{2}(n+\tau m) \in \mathbf{C}^g$ for some $n,\, m \in \{0,\,1\}^g$. Let $\tilde{b}_\tau = \frac{1}{4}(n+\tau m)$, then $\tilde{b}$ represents a point $b_\tau \in X_\tau$ with $2b_\tau = a_\tau$. We have seen that $\Theta_\tau(b_\tau)$ lies in an eigenspace of a, let's call that one $\mathbf{P}V_a^+$. Then (X_τ, a_τ) satisfies the Schottky-Jung relations iff

$$\Theta_\tau(b_\tau) \in \overline{\Theta(\mathbf{H}_{g-1})} \qquad (\subset \mathbf{P}V_a^+).$$

The ideal of the projective variety $\overline{\Theta(\mathbf{H}_g)}$ $(\subset \mathbf{P}V)$ will be denoted by I_g, so I_g $(\subset \mathbf{C}[\ldots, X_\sigma, \ldots]_{\sigma\in\{0,1\}^g})$ contains all the homogeous polynomials which are zero on that variety. Using the Heisenberg group actions, I_{g-1} can be identified with the ideal I_{g-1}^{a+} of $\overline{\Theta^{a+}(\mathbf{H}_{g-1})} \subset \mathbf{P}V_a^+$ for any $a \in H/\mathbf{C}^*$. Then (X_τ, a_τ) satisfies the Schottky-Jung relations if and only if

$$\sigma_a(P)(\tau) := P(\Theta_\tau(b_\tau)) = 0 \qquad (\forall P \in I_{g-1}^{a+},\ P \text{ homogeneous}).$$

For every $a \in H/\mathbf{C}^*$ and every homogeneous $P \in I_{g-1}$ we get a holomorphic function $\sigma_a(P) : \mathbf{H}_g \longrightarrow \mathbf{C}$. Since the coordinate functions of $\tau \mapsto \Theta_\tau(b_\tau)$ are modular forms of 'half integral' weight, the holomorphic functions $\sigma_a(P)$ are also modular forms. So we see that the Schottky loci can be defined by modular forms and that

$$S_g^{small} = \{[\tau] \in A_g = \Gamma_g\backslash\mathbf{H}_g :\ \sigma_a(P)(\tau) = 0\ \forall a \in H/\mathbf{C}^* - \{0\} \text{ and all homogeneous } P \in I_{g-1}\}.$$

In general we know very little about I_g. In the first non-trivial case, $g = 3$, we know that I_3 is generated by one polynomial F of degree 16. This implies that in genus 4 we get a modular form $\sigma_a(F)$ of weight 8 for each $a \in H/\mathbf{C}^*$. Schottky already proved that all these modular forms are the same and that they are not identically zero. Much later Igusa and Freitag proved that the zero locus of this modular form is exactly J_4. It is known that I_4 contains elements of degree 32.

5.5.2 Proof of Theorem 5.4.6. We give a rough sketch of the proof of Theorem 5.4.6. In fact, to simplify matters we will assume that we have an universal family of abelian varieties over A_g and this family as well as A_g are smooth(!). To justify the arguments given here one has to use level structures. We will also make some other simplifying assumptions but we believe that we still convey the main idea of the proof.

Since S_g^{small} is a quasi-projective variety, it is a finite union of irreducible components:

$$S_g^{small} = Z_1 \cup Z_2 \cup \ldots \cup Z_n.$$

For simplicity we write

$$J_g := \overline{j(M_g)}\ (\subset A_g).$$

As $J_g \subset S_g^{small}$ and J_g is irreducible (it is the closure of the image of the Teichmüller space) it must be contained in at least one of the Z_i, let's say $J_g \subset Z_1$. If $\dim Z_1 = 3g-3$ $(= \dim J_g)$, then, since both are irreducible, $J_g = Z_1$. So the theorem follows if we prove that $\dim Z_1 = 3g - 3$. We will use induction on g to prove this, the case $g = 2$ (or $g = 3$) being trivial.

We recall that the Satake compactication $\overline{A}_g$ is the disjoint union of A_g and $\overline{A}_{g-1}$. It is well-known that $\overline{J_g}$, the closure of J_g in $\overline{A}_g$, intersects $\overline{A}_{g-1}$ in $\overline{J}_{g-1}$. It is not hard to show that something similar happens with the small Schottky locus:

$$\overline{J_g} \cap \overline{A}_{g-1} = \overline{J}_{g-1}, \qquad \overline{S_g^{small}} \cap \overline{A}_{g-1} \subset \overline{S_{g-1}^{small}}.$$

Let $Z_1' := \overline{Z_1} \cap \overline{A}_{g-1}$, then $Z_1' \subset \overline{S_{g-1}^{small}}$. Since $J_g \subset Z_1$ we get $J_{g-1} \subset Z_1' \subset \overline{S_{g-1}^{small}}$. The induction hypothesis that J_{g-1} is an irreducible component of S_{g-1}^{small} implies that $\overline{J}_{g-1}$ is an irreducible component of Z_1'. Unfortunately, the codimension of $\overline{A}_{g-1}$ in $\overline{A}_g$ is very large (it is $\frac{1}{2}g(g+1) - \frac{1}{2}(g-1)g = g$). Therefore we cannot get a good estimate of $\dim Z_1$ from this fact.

5.5.3 Igusa's compactification. We consider another compactification of A_g, introduced by Igusa. Let

$$\beta : \tilde{A}_g \longrightarrow \overline{A}_g$$

be the blow up of $\overline{A}_g$ along its boundary $\overline{A}_{g-1}$. The inverse image $\beta^{-1}(\overline{A}_{g-1})$ of $\overline{A}_{g-1}$ is a divisor on $\tilde{A}_g$. Let $\tilde{Z}_1$ be the closure of Z_1 in $\tilde{A}_g$ and let $\tilde{Z}_1'$ be an irreducible component of $\tilde{Z}_1 \cap \beta^{-1}(A_{g-1})$ mapping onto the irreducible component J_{g-1} of Z_1'.

Assuming there are smooth points of $\tilde{A}_g$ in $\tilde{Z}_1'$, we get:

$$\dim Z_1 = \dim \tilde{Z}_1 = \dim(\tilde{Z}_1') + 1.$$

If we can show that the map

$$\beta_{|\tilde{Z}_1} : \tilde{Z}_1' \longrightarrow J_{g-1}$$

has a fiber of dimension ≤ 2, then it follows that $\dim \tilde{Z}_1' \leq (3(g-1)-3)+2 = 3g-4$ and thus $\dim Z_1 = 3g-4+1 = 3g-3$ as desired.

The inverse image of the open subset $A_{g-1} \subset \overline{A}_{g-1}$ under the map β is the 'universal family of abelian varieties' over A_{g-1} (note we simplify here). Thus we may identify $\beta^{-1}([\pi]) = X_\pi$. Since $\tilde{Z}_1{}' \subset \tilde{S}^g_{small}$, the closure of the small Schotky locus in $\tilde{A}_g$, it suffices to show that $\dim(\tilde{S}^g_{small} \cap X_\pi) \leq 2$.

Recall that S_g^{small} is defined by the modular forms $\sigma_a(P)$ with $P \in I_{g-1}$. We need to know the intersection of the (closure in $\tilde{A}_g$) of the zero locus of $\sigma_a(P)$ with $\beta^{-1}(\pi) = X_\pi$.

For this we consider a period matrix:

$$\tau = \begin{pmatrix} \pi & z \\ z & \tau_1 \end{pmatrix} \ (\in \mathbf{H}_g), \qquad \pi \in \mathbf{H}_{g-1},\ z \in \mathbf{C}^{g-1},\ \tau_1 \in \mathbf{H}_1.$$

(for any $\pi,\ z$ we can find τ_1 (with $Im\tau_1 >> 0$) such that $\tau \in \mathbf{H}_g$). This matrix τ moves to the boundary point $\pi \in A_{g-1}$ if we let $Im\tau_1 \to +\infty$ (equivalently, $q := e^{2\pi i \tau_1} \to 0$, here $\pi = 3.14\ldots$). The modular form $\sigma_a(P)$ has a q-expansion:

$$\sigma_a(P)(\tau) = \sum_{n \in \mathbf{Z}_{\geq 0}} P_n(\pi, z) q^n.$$

The functions $z \mapsto P_n(\pi, z)$ are theta functions on X_π. The intersection of the zero locus of $\sigma_a(P)$ with X_π is given by zero's of $P_k(\pi, z)$ where k is the smallest integer for which $P_n(\pi, z)$ is not identically zero.

The coordinates $\theta_\sigma(\tau, b_\tau)$ of $\Theta_\tau(b_\tau)$ have the q-expansion:

$$\theta_\sigma(\tau, b_\tau) = \theta_{\bar{\sigma}}(\pi, 0) + \theta_{\bar{\sigma}}(\pi, z)q + H.O.T. \qquad \bar{\sigma} = (\sigma_1, \ldots, \sigma_{g-1})$$

when $\sigma = (\sigma_1, \ldots, \sigma_g)$. Since P is a polynomial, we get the following formula:

$$\sigma_a(P)(\tau) = P(\ldots, \theta_{\bar\sigma}(\pi, 0), \ldots) + q \sum_{\bar\sigma} \frac{\partial P}{\partial X_{\bar\sigma}}(\ldots, \theta_{\bar\sigma}(\pi, 0), \ldots)\theta_{\bar\sigma}(\pi, z) + H.O.T.$$

Since the polynomials $P \in I_{g-1}$ are zero on the image of $\mathbf{H}_{g-1}$ we have $P(\ldots, \theta_{\bar\sigma}(\pi, 0), \ldots) = 0$. We have seen in 4.3.4 that $\sum_{\bar\sigma} \frac{\partial P}{\partial X_{\bar\sigma}}(\ldots, \theta_{\bar\sigma}(\pi, 0), \ldots)\theta_{\bar\sigma}(\pi, z)$ is in Γ_{00} and moreover, if $\Theta(\pi)$ is a smooth point on $\overline{\Theta(\mathbf{H}_{g-1})}$ then Γ_{00} is spanned by these functions (where P runs over I_{g-1}).

Putting everything together we get:

$$\tilde{S}_g^{small} \cap \beta^{-1}(\pi) \subset F_\pi \ = \{z \in X_\pi :\ \theta(z) = 0 \quad \forall \theta \in \Gamma_{00}\}.$$

Thus it suffices to show that for the period matrix π of a genus $g-1$ curve we have $\dim F_\pi \leq 2$. Fortunately, this was proved by Welters and thus the proof is complete.

5.5.4 Remark. One can 'see' why $J_g \cap \beta^{-1}([J(C)])$ is $C - C \subset J(C)$ $(= \beta^{-1}([J(C)]))$. If one degenerates a genus g curve to a curve with a node, the normalization C of the nodal curve has genus $g-1$ and the inverse image of the node are two points p, q on C. Following the moduli point of the genus g curve we arrive at the point $[J(C)] \in A_{g-1}$ in the Satake compactification $\overline{A}_g$ and at the point $p - q \in F_{J(C)} \subset J(C) = \beta^{-1}([J(C)])$ in Igusa's compactification $\tilde{A}_g$.

5.5.5 Remark. Recall that in case $g = 5$ the locus W of intermediate Jacobians of cubic threefolds are contained in S_5^{big}. Its closure $\overline{W}$ in the Satake compactification intersects the boundary $\overline{A}_4$ in $\overline{j(M_4)}$. Consider now the closure $\tilde{W} \subset (\tilde{A}_5)$ of this locus in Igusa's compactification and the blow down map $\beta : \tilde{W} \to \overline{W}$. Let $[JC] \in j(M_4)$ be a general Jacobian, then one finds that $\beta^{-1}([JC]) \cap \tilde{W}$ consists of the points $\pm(g^1_3 - h^1_3)$ which are exactly the exceptional points in F_{JC}, the subvariety defined by Γ_{00} (see 4.3.6).

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of Algebraic Curves, Vol. 1. Springer-Verlag (1985).

[B3] A. Beauville, *Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans* $\mathbf{P}^4$. LNM 947, Springer Verlag (1982) 190–208.

[B] A. Beauville, *Le problème de Schottky et la conjecture de Novikov.*, Exp. **675** du sém. Bourbaki, Astèrisque 152-153 (1988), 101–112.

[BD] A. Beauville, O. Debarre, *Sur les fonctions theta du second ordre.*, Proceedings: Arithmetic of Complex Manifolds, LNM **1399** (1989), 27–39.

[C] C. H. Clemens, *A Scrapbook of Complex Curve Theory.* Plenum Press, New York and London (1980).

[CGV] M. Cornalba, X. Gomez-Mont, A. Verjovsky, *Lectures on Riemann Surfaces.* World Scientific (1989).

[D1] R. Donagi, *Non-Jacobians in the Schottky loci.* Ann. of Math. **126** (1987), 193–217.

[D2] R. Donagi, *The Schottky problem.* Proceedings: Theory of Moduli, LNM 1337, Springer-Verlag (1988), 84–137.

[Fa] J. D. Fay, *Theta functions on Riemann surfaces.* LNM **352** (1972).

[Fr] E. Freitag, *Die irredcibilität der Schottky relation.* Archiv der Mathematik **40** (1983), 255-259.

[vdG] G. van der Geer, *The Schottky Problem.*, Arbeitstagung Bonn 1984. Springer Verlag 1985.

[vG] B. van Geemen, *Siegel modular forms vanishing on the moduli space of curves.* Invent. Math. **78** (1984) 329–349.

[GG] B. van Geemen, G. van der Geer, *Kummer varieties and the moduli spaces of abelian varieties.* Am. J. Math. **108** (1986) 615–642.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* J. Wiley & Sons (1978).

[Ig1] J-I. Igusa, *Theta Functions.* Grundlehren 194, Springer-Verlag (1972).

[Ig2] J-I. Igusa, *On the irreducibility of Schottky's divisor*, J. Fac. Sci. Tokyo **28** (1981) 531–545.

[Iz] E. Izadi, *The geometric structure of $\mathcal{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors.* J. reine angew. Math. **462** (1995), 93–158.

[K] G. R. Kempf, *Complex Abelian Varieties and Theta Functions.* Universitext, Springer-Verlag (1990).

[LB] H. Lange, Ch. Birkenhake, *Complex Abelian Varieties.* Grundlehren 302, Springer-Verlag (1992).

[M1] D. Mumford, *Abelian Varieties.* Oxford University Press (1970).

[M2] D. Mumford, *Curves and Their Jacobians.* The Univ. of Michigan Press (1975).

[M3] D. Mumford, *Tata Lectures on Theta II.* Progr. in Math. **43**, Birkhäuser (1984).

[V] R. Varley

[W] Welters, *The surface C-C in Jacobi varieties and second order theta functions.* Acta Math. **157** (1986) 1–22.

Dipartimento di Matematica,
Università di Torino,
Via Carlo Alberto 10,
10123 Torino,
Italy.
geemen@dm.unito.it