A non-selfdual automorphic representation of GL_3 and a Galois representation

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Abstract *The Langlands philosophy contemplates the relation between automorphic representations and Galois representations. A particularly interesting case is that of the non-selfdual automorphic representations of GL_3. Clozel conjectured that the L-functions of certain of these are equal to L-functions of Galois representations.*

Here we announce that we found an example of such an automorphic representation and of a Galois representation which appear to have the same L-functions (for a more precise statement see Prop. 3.6).

1 Introduction

1.1 There is a well known procedure which associates to any cusp form f on the congruence subgroup $\Gamma_0(N)$ of $SL(2,Z)$, which is an eigenform for the Hecke algebra, a representation $\sigma(f)$ of $Gal(\overline{Q}/Q)$ on a two dimensional vector space over a finite extension of Q_l (for any prime number l) ([D]). Moreover, one has an equality of L-series:

$$L(f,s) = L(\sigma(f),s).$$

In case the weight of f is 2, this Galois representation is in $H^1(X_0(N),Q_l)$, the first etale cohomology group of the modular curve $X_0(N)$.

The notion of cusp form has been generalized to that of 'algebraic' cuspidal automorphic representation, these can be defined for any reductive algebraic group G over a number field. In case there are Shimura varieties $M(G)$ associated to G one tries to associate to such an automorphic representation π of G an l-adic sheaf F_π on M and a Galois representation $\sigma(\pi)$ on $H^*(M,F_\pi)$, in such a way that the L-functions of π and $\sigma(\pi)$ coincide.

In case there is no Shimura variety associated with G, and this happens for example if $G = GL_3$, there seems to be no method (even conjecturally) that might associate Galois representations to cuspidal automorphic representations of G except for special classes (like the selfdual representations of GL_n, cf. [C2]). Within the Langlands' philosophy however, one does expect such Galois representations to exist, for an explicit conjecture by Clozel, see [C1] conj. 4.5. Some 10 years ago Ash already tried to find examples, lack of computer power at that time probably prevented him from finding the example below.

1991 *Mathematics Subject Classification.* Primary 11R40, 14G10, 22E55 Secondary 11R39.

1.2 The result which we would like to announce here is that there exists a cuspidal automorphic representation π_u of $GL_{3,Q}$ (not selfdual) such that the product of the local L-factor of π_u and that of its contragredient coincides with the local L-factor of a (compatible system of λ-adic) Galois representation(s) σ for all primes p, $3 \leq p \leq 67$ (cf. Prop. 3.6). (Using faster programs/computers and/or more patience one could try to verify the equality for more primes.)

1.3 The representation π_u corresponds to a cohomology class $u \in H^3(\Gamma_0(128), C)$, with now $\Gamma_0(N)$ the subgroup of matrices in $SL_3(Z)$ with $a_{21} \equiv a_{31} \equiv 0 \bmod N$. In the next section we explain how the local L-factors of π_u are computed.

In case there is a λ-adic Galois representation corresponding to π_u one expects it to be unramified outside $p = 2$ (in general: unramified outside the primes dividing N) and l, with $\lambda | l$. To construct the compatible system of Galois representations we search for suitable subspaces V_l in the etale cohomology of a suitable algebraic variety S defined over Q, with good reduction outside $p = 2$. These subspaces should be motivically defined (that is, should be cut out by correspondences). In particular one has a $V_{\infty,C} \subset H^*(S, C)$ and one can define its Hodge numbers $h^{p,q} := \dim V_{\infty,C} \cap H^{p,q}(S, C)$. These Hodge numbers ought to correspond to the infinity type of π_u, in this case they should be: $h^{2,0} = h^{1,1} = h^{2,0} = 1$ (cf. [AS], p.216).

This suggests looking at the H^2 of surfaces. Since the coefficients of the local L-factors are in $Z[i]$, we will look for 6 dimensional spaces V_l, with Hodge numbers $h^{2,0} = h^{1,1} = h^{0,2} = 2$. Moreover we want an automorphism ϕ of order 4 on S, defined over Q (so on H^2, ϕ^* commutes with the action of $Gal(\overline{Q}/Q)$) and which has the eigenvalues i and $-i$, each with multiplicity 3, on $V_l \otimes_{Q_l} \overline{Q}_l$. Then we get 3 dimensional $Gal(\overline{Q}/Q)$-representations of the desired type on each of the two eigenspaces. The construction of a suitable S and ϕ is given in section 3. In the last section we discuss variants of our constructions and related questions.

Finally we would like to mention that in [APT] a (unique) Galois representation $\rho : Gal(\overline{Q}/Q) \to GL(F_3)$ is constructed which has the same L-factors (mod $\sqrt{-3}$) for small primes as a certain automorphic representation on GL_3, this provided the first evidence that there might be Galois representations on λ-adic vector spaces associated to such automorphic representations.

1.4 We would like to thank A. Ash for constant advice and encouragement, L. Clozel for suggesting the problem to one of us, and W. van der Kallen and A. de Meijer for their help in mastering the computer.

2 The automorphic representation

2.1 The automorphic representation of GL_3 we consider is defined by a cuspidal cohomology class in $H^3(\Gamma_0(128), C)$. The subspace of all cuspidal classes is denoted by $H^3_!(\Gamma_0(N), C)$ (cf. [AGG], section 2 and the references given there). There is a surjective map ([AGG], Lemma 3.5):

$$H_3(\Gamma_0(N), C) \longrightarrow H^3_!(\Gamma_0(N), C)^*$$

(with a 'known' kernel). The space $H_3(\Gamma_0(N), C)$ can be computed explicitly using results of [AGG]:

2.2 Proposition. For any integer N, we define a complex vector space

$$W(\Gamma_0(N)) := \left\{ P^2(Z/NZ) \xrightarrow{f} C : \quad \begin{array}{c} f(x,y,z) = f(z,x,y) = -f(-y,x,z), \\ f(x,y,z) + f(-y,x-y,z) + f(y-x,-x,z) = 0 \end{array} \right\}.$$

Then there is a natural isomorphism

$$\Phi : W(\Gamma_0(N)) \longrightarrow H_3(\Gamma_0(N), C).$$

2.3 The (commutative) Hecke algebra $\mathcal{T}_N$ is generated by linear maps

$$E_p,\ F_p : H^3(\Gamma_0(N), C) \longrightarrow H^3(\Gamma_0(N), C)$$

for primes p, $(p \not| N)$ ([AGG], Prop. 4.1). The action on $H_3(\Gamma_0(N), C)$ of the adjoint of a Hecke operator can be computed, using Prop. 2.2, with modular symbols ([AGG], section 4, 6B).

2.4 Let $u \in H^3_!(\Gamma_0(N), C)$ be an eigenclass for all Hecke operators, and define complex numbers (actually algebraic integers) by

$$E_p u = a_p u, \qquad \text{then} \quad L_p(\pi_u, s) = (1 - a_p p^{-s} + \overline{a_p} p^{1-2s} - p^{3-3s})^{-1}$$

is the L-factor at p of the cuspidal automorphic representation π_u corresponding to u (and $F_p u = \overline{a_p} u$).

2.5 By explicit (computer) computation we determined in this way the existence of a unique (up to scalar multiples) pair of eigenclasses $u, v \in H^3(\Gamma_0(128), C)$ which satisfied:

$$E_p u = a_p u, \quad E_p v = \overline{a_p} v, \qquad Q(\ldots, a_p, \ldots) = Q(i),$$

the automorphic representation π_v associated to v is the contragredient of π_u. The explicit values we found are listed below (the a_p are all in $Z[2i]$):

a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}	a_{23}	a_{29}
$1+2i$	$-1-4i$	$1+4i$	$-7-10i$	$-1+4i$	7	$1-14i$	$17-4i$	$-9-12i$

a_{31}	a_{37}	a_{41}	a_{43}	a_{47}	a_{53}	a_{59}	a_{61}	a_{67}
1	$-25+28i$	-5	$-7+30i$	$17+40i$	$23-20i$	$-39+22i$	$63+20i$	$65-22i$

(π_u is not selfdual since there is no Dirichlet character χ with $a_p = \chi(p)\overline{a_p}$ for all p.)

3 The Galois representation

3.1 A minimal surface S with $h^{2,0} = p_g = 2$ has a (rational) canonical map $V \to P^1$, if the fibers are elliptic curves it is actually a morphism, in that case one has $h^1(S) = 0$, $h^2(S) = 34$. If S is such a fibration with a section, then there is an involution on S, fiberwise -1 on the elliptic curve, and S modulo that involution is a rational surface. Examples might thus be found among double covers of P^2 ramified along a curve of degree at least 8 (to get $h^{2,0} > 1$) and suitable singularities (to get $h^{2,0} < 3$).

The following surfaces were first studied by Ash and Grayson (actually we have a slightly modified form). Let S_a be the (projective) minimal model S_a of the 2:1 cover of P^2 defined by the (affine) equation:

$$t^2 = xy(x^2-1)(y^2-1)(x^2-y^2+axy) \qquad (a \in Z - \{0\}).$$

Over $\overline{Q}$, the branch curve B is a union of 8 lines, it has 16 double points, 2 triple points $((1:0:0),\ (0:1:0))$ and one fourfold point $(P := (0:0:1))$. The fourfold point imposes one adjunction condition, so $h^{2,0}(V) = 2$, and the (rational) canonical map is given by the pencil of lines through P. The fibers of the canonical map are thus elliptic curves, so $h^2(S_a) = 34$, $h^1(S_a) = 0$.

3.2 The Neron-Severi group of S_a contains, over $\overline{Q}$, a 28 dimensional space $N_{\overline{Q}}$ spanned by the following divisors: the pull-back of a line on P^2, the 16 P^1's mapping to the 16 double points of B, the $2 \cdot 4$ P^1's mapping to the 2 triple points, the fiber E_P over the fourfold point, a P^1 mapping to the diagonal $x = y$ and a P^1 mapping to the 'anti-diagonal' $x = -y$.

For any l, we define a $Gal(\overline{Q}/Q)$-subrepresentation of $H^2(S_{a,\overline{Q}}, Q_l)$ by:

$$N_{Q_l} := Image(N_{\overline{Q}} \longrightarrow H^2(S_{a,\overline{Q}}, Q_l)).$$

Let T_{Q_l} be the orthogonal complement of N_{Q_l} w.r.t. the intersection form, then:

$$H^2(S_{a,\overline{Q}}, Q_l) = T_{Q_l} \oplus N_{Q_l},$$

a direct sum of $Gal(\overline{Q}/Q)$ representations, in particular $\dim T_{Q_l} = 6$.

The map:

$$(x,y,t) \mapsto (y,-x,t) \qquad \text{induces} \quad \phi : S_a \to S_a,$$

an automorphism of order 4, defined over Q and it has eigenvalues $i, -i$ on $H^{2,0}(S_a)$.

3.3 Using the basis of $N_{\overline{Q}}$ and $h^1 = 0$ it is easy to find a formula for the trace of Frobenius acting on T_{Q_l} using the Lefshetz trace formula. Let $E_\infty : w^2 = v(v^2 + av - 1)$, it is the curve in S_a over the line at infinity in P^2, moreover $E_P \cong_Q E_\infty$. Then we have:

3.4 Proposition. Let $F_p \in Gal(\overline{Q}/Q)$ be a Frobenius element at p, with $p \not| 2a(a^2+4)$. Then:

$$Trace(F_p^n | T_{Q_l}) = N_q(S_a) + 2N_q(E_\infty) - q^2 - 2q(1+\delta^n),$$

with $q = p^n$, the Legendre symbol $\delta := \left(\frac{a}{p}\right)$ and

$$\begin{aligned} N_q(S_a) &:= \sharp\{(x,y,t) \in F_q^3 : \ t^2 = xy(x^2-1)(y^2-1)(x^2-y^2+axy)\ \}, \\ N_q(E_\infty) &:= \sharp\{(v,w) \in F_q^2 : \ w^2 = v(v^2+av-1)\ \}. \end{aligned}$$

3.5 To determine the eigenvalue polynomial of F_p on N_{Q_l} one would have to compute the number of points over F_{p^i} for $i = 1, \dots, 6$. However, we have that if α is an eigenvalue of F_p, then so is $\overline{\alpha}$ and moreover $Det(F_p|T_{Q_l}) = p^6$. Therefore the eigenvalue polynomial of F_p looks like:

$$H_p = X^6 - c_1X^5 + c_2X^4 - c_3X^3 + p^2c_2X^2 - p^4c_1X + p^6.$$

Since the Galois representation on T_{Q_l} is reducible (after adjoining an $i = \sqrt{-1}$ to Q_l if necessary), say $T_{Q_l} = V_1 \oplus V_2$, we obtain a one dimensional Galois representation on $\wedge^3 V_1$, which is the cube of the cyclotomic character times a $Q(i)$-valued Dirichlet character χ, and χ is unramified outside $2a(a^2+4)$. Computing the c_i for some small primes suffices to determine χ, in the case $a = 2$ we considered we had $\chi(p) = 1$ iff $p \equiv 1,\ 3 \bmod 8$, and $\chi(p) = -1$ for other $p > 2$. The eigenvalue polynomial of F_p then factors as

$$H_p = (X^3 - \chi(p)b_pX^2 + p\overline{b_p}X - \chi(p)p^3)(X^3 - \chi(p)\overline{b_p}X^2 + pb_pX - \chi(p)p^3).$$

Thus the number of points over F_p and F_{p^2} determines the b_p's for larger primes. For each p we then have a set $\{b_p,\ \overline{b_p}\}$, but we don't know which cubic factor of H_p is (minus) the eigenvalue polynomial on V_1.

To get L-factors similar to those of an automorphic representation of GL_3, we twist the Galois representation on T_{Q_l} by the non-trivial one dimensional representation

$$\epsilon : Gal(\overline{Q}/Q) \to Gal(Q(\sqrt{-2})/Q) \to \{\pm 1\}, \qquad \text{thus}: \quad \epsilon(F_p) = \chi(p).$$

The eigenvalue polynomial of F_p on $T_{Q_l} \otimes \epsilon$ (note the abuse of notation) thus differs from H_p by replacing $\chi(p)$ by 1. (This twisted representation occurs in the H^2 of the surface $S_a^\sharp$ whose equation is obtained by replacing t^2 by $-2t^2$ in the equation defining S_a.) The 6 dimensional (reducible) Galois representation to be considered is then:

$$\sigma : Gal(\overline{Q}/Q) \longrightarrow GL(T_{Q_l} \otimes \epsilon).$$

3.6 Proposition. For the surface S_2 (so $a = 2$) and the b_p's defined as above, we have:

$$a_p \in \{b_p,\ \overline{b_p}\} \qquad \text{for} \quad 3 \le p \le 67.$$

with a_p the eigenvalue of the Hecke operator E_p on the non-selfdual cohomology class u on $\Gamma_0(128)$. In particular, for all primes $l \neq p$:

$$L_p(\sigma, s) = L_p(\pi_u, s)L_p(\pi_v, s) \qquad \text{for} \quad 3 \le p \le 67.$$

4 Generalizations and Problems

4.1 Besides the surfaces S_a we have various other families of surfaces which have sub Galois representations of the desired type in H^2. For example, let $\mathcal{E}_a$ be the Neron model of the elliptic surface

$$\mathcal{E}_a: \quad Y^2 = X(X^2 + 2a(t^2+1)X + 1).$$

It has 5 singular fibers, one of type I_8 and 4 of type I_1 and it has a section of infinite order. Next one pulls back $\mathcal{E}$ along a 3:1 or 4:1 Galois cover $P^1 \to P^1_t$ branching over two of the I_1 fibers. The Neron-Severi group of the pull back has rank (at least) 28, and we can again define T_{Q_l}'s as before. In this case however, the representations we find seem to be selfdual (we found $b_p = \bar{b}_p$ for small p, if true for all p this would imply $V_1 \cong V_2$), and in one case we could actually find a 2 dimensional Galois representation ρ such that $Sym^2(\rho)$ has the same b_p's for small p's.

4.2 Another example is provided by the Neron models of the elliptic surfaces

$$\mathcal{E}'_a: \quad y^2 = x^3 + a(x + t^2(t-1))^2.$$

These have one I_6, one I_3 and three I_1 fibres as well as a section of infinite order. Pulling them back as in the previous example one again obtains 6 dimensional Galois representations, which split in two 3 dimensional pieces, which now are not selfdual in general (actually for the 4:1 cover, one has to modify the construction of T since the surface itself has $h^{2,0} = 3$ in that case). We hope to relate also these surfaces to automorphic representations of GL_3.

4.3 To an elliptic curve over Q one associates a two dimensional Galois representation σ_E on $H^1(E_{\overline{Q}}, Q_l)$. The Taniyama-Weil conjecture asserts that $\sigma_E = \sigma(f)$ for a suitable cusp form f of weight 2. Since our surface S occurs in a one dimensional family it is tempting to ask if for every S_a with $a \in Q$ one can find an automorphic representation π_a of GL_3 such that $L(T_a, s) = L(\pi_a, s)L(\pi_a^*, s)$.

4.4 The number of Galois representations with a given conductor and a given dimension is finite, and in case the representations are two dimensional good criteria are known for two such representations to be isomorphic (see [L] and the references given there). One could try to strengthen Prop. 3.6 with the statement: If there is a Galois representation associated to π_u, then it must be isomorphic to an irreducible component of σ. We hope to return to this issue in our paper.

4.5 One can also try to generalize Serre's work and conjectures on mod p Galois representations and mod p modular forms from GL_2 to GL_3. Promising experimental evidence has been found by Ash and McConnell, [AM].

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