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On the Geometry and Arithmetic of Some Siegel Modular Threefolds

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0. INTRODUCTION

In this paper we consider some examples of Siegel modular 3-folds. These examples are all associated to certain subgroups of $Sp_4(\mathbb{Z})$, containing the principal congruence subgroup $\Gamma(8)$ with finite index. More precisely we consider a subgroup, $\Gamma(2, 4, 8)$, of index 2 in the theta group $\Gamma(4, 8)$ (the exact definitions are given in section 1). We use theta constants to define an embedding of $\mathscr{A}(2, 4, 8) := \mathfrak{S}_2/\Gamma(2, 4, 8)$ into $\mathbb{P}^{13}$, where $\mathfrak{S}_2 = \{\tau \in M_2(\mathbb{C}) \mid \tau = {}^t\tau \text{ and } \operatorname{Im}\tau \text{ is positive definite}\}$. Riemann's theta relations show that the closure of the image is a complete intersection of 10 quadrics. This complete intersection, however is rather complicated, for instance any resolution has $h^{30} = 2283$ (see [vG-vS]). Instead we study three types of quotients of $\mathscr{A}(2, 4, 8)$ (denoted X, Y, Z). The variety $\mathscr{A}(2, 4, 8)$ is a quotient of $\mathscr{A}(2, 4, 8)$ (denoted X, Y, Z). The variety $\mathscr{A}(2, 4, 8)$ is a ramified Galois cover of $\mathscr{A}(2, 4)$ with group $(\mathbb{Z}/2)^{10}$. The key observation is that the Satake compactification of $\mathscr{A}(2, 4)$ is $\mathbb{P}^3$ thus each of the three types of quotients can be viewed as a ramified Galois cover of $\mathbb{P}^3$. Again the Riemann theta relations allow us to realize them as complete intersections, this time of four quadrics in $\mathbb{P}^7$.

Our main goal is to determine the Betti and Hodge numbers and the L-functions associated to the Galois representations on H^3_l of suitable resolutions of these varieties. By a careful study of the rationality properties of the exceptional fibers in the resolutions and using a computer we count the number of points over various finite fields. Using the Weil conjectures allow us to compute the Betti numbers and, using results of Faltings, Serre and Livne, to determine the L-functions.

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The paper is organized as follows: In Section 1 certain results, concerning theta groups and projective embeddings by theta constants, are proved. These results are used in section 2 to associate to any 6-tuple of distinct even theta characteristics a complete intersection of four quadrics in $\mathbb{P}^7$. Under the action of the Siegel modular group $Sp_4(\mathbb{Z})$ there are 3 orbits of such 6-tuples. It follows from the transformation laws for theta constants that the varieties associated to the 6-tuples within an orbit are all isomorphic over the cyclotomic field $\mathbb{Q}(\zeta_8)$ and thus it suffices to consider one 6-tuple from each orbit.

In Section 2 the 3-folds associated to the orbits are studied.

The first orbit we consider consists of 15 6-tuples, characterized by the sum of the theta characteristics being 0. The associated 3-fold, X, has 96 ordinary double points and H^3 of the blow-up, X', has rank 2 with $h^{03}=1$. We prove that the L-function of H^3_l is equal to $L(g, s)$, where g is the unique new-form of weight 4 on $\Gamma_0(8) \subset SL_2(\mathbb{Z})$.

On the other hand the holomorphic 3-form on X' corresponds to a Siegel modular form, φ of weight 3. In fact it is the product of the six theta constants corresponding to the 6-tuple of theta characteristics defining X and is a cusp form for the principal congruence subgroup $\Gamma(4) \subset Sp_4(\mathbb{Z})$. In an appendix we sketch a computation of the Andrianov L-function of this form, using Shintani's and Oda's results on the Saito–Kurokawa lift [Sh, O]. The main result is that the form is in fact the Saito–Kurokawa lift of the elliptic modular form g; this essentially comes down to proving that a certain period integral is non-vanishing. We find that $L(\varphi, s) = \zeta(s-1)\, L(g, s)\, \zeta(s-2)$. This relation between the L-functions are predicted by the general conjecture on the Galois representation associated to a Saito–Kurokawa lift.

We note that the Galois representation associated to the form g also occurs in H^3_l of the elliptic modular threefold of $\Gamma_0(8)$. The Tate conjecture predicts that the isomorphism between the Galois representations is induced by an algebraic correspondence, i.e., an algebraic cycle on the product of the two varieties. In fact J. Stienstra has explicitly exhibited such a correspondence and later a modular interpretation of this correspondence has been given by Ekedahl and the first author [E-vG].

The second orbit consists of 180 6-tuples which are characterized as being the disjoint union of an azygeous and a syzygeous 3-tuple. The 6-fold products of theta constants are cusp forms. The singular locus of the associated threefold, Y, consists of 16 ordinary double points and 4 conics intersecting transversally configured in a square. We construct a resolution, Y', and show by similar methods that $h^3(Y') = 4$ and that the L-function of H^3_l is the product of two Hecke L-series. In this case the Tate conjecture predicts the correspondence between Y' and the triple product of an elliptic curve with CM by the field $\mathbb{Q}(i)$. We explicitly construct a correspondence inducing an isomorphism of the $H^{3,0}$'s. Though we were not able to determine the Andrianov L-function of the associated Siegel modular form our computations indicate that it is a Yoshida lifting [Yo].

Finally we consider the orbit consisting of the remaining 15 6-tuples. In this case the associated products of theta constants are not cusp forms. The corresponding threefold, Z, has two non-intersecting lines as its singular locus and is birationally equivalent to $V \times \mathbb{P}^1$ where V is the quartic Fermat surface. Hence its resolution does not have any holomorphic 3-forms. In contrast with the other two examples; however, it has a holomorphic 2-form, $h^{20} = 1$. Using results of Weissauer [We] we determine the Galois representation on H^2.

Although the examples considered in this paper are of a very special nature, we feel that they do provide an interesting testing ground for conjectures about the arithmetic of Siegel modular 3-folds. Also these examples point to interesting correspondences between Siegel modular 3-folds and other types of modular varieties.

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1. Theta Groups

Let A be an abelian variety of dimension g and let L be a symmetric line bundle, i.e., $L \simeq i^*L$ where i is the inversion on A, which defines a principal polarization. The line bundle $L^{\otimes 2}$ is canonically determined by the polarization (while L itself is only determined up to translation by a point of order 2). Let $\mathscr{A}_g(2n, 4n)$ be the moduli space of principally polarized abelian varieties, ppav's for short, with a symmetric theta structure of level $2n$ (for definitions see [Mu1]). A theta structure t of level $2n$ on A defines an ordered basis of $H^0(A, L^{2n})$, $\{\vartheta_{\beta,2n}\}$ with $\beta \in (\mathbb{Z}/2n)^g$. As $i^*\vartheta_{\beta,2n} = \vartheta_{-\beta,2n}$, the subspace of even sections, $H^0(A, L^{2n})^+$, has dimension $2^{g-1}(n^g + 1)$.

The exist canonical morphisms

$$\Theta_{2n} : \mathscr{A}_g(2n, 4n) \to \mathbb{P}^m$$

with $m = 2^{g-1}(n^g + 1) - 1$, defined by the even theta constants of level $2n$:

$$\Theta_{2n}(A, t) = (\cdots : \vartheta_{\beta,2n}(0) + \vartheta_{-\beta,2n}(0) : \cdots)$$

Let $\mathscr{F}_{2n}$ denote the pull-back to $\mathscr{A}_g(2n, 4n)$ of $\mathcal{O}(1)$ by this morphism.

Over $\mathbb{C}$, $\mathscr{A}_g(2n, 4n) \simeq \mathfrak{S}_g/\Gamma(2n, 4n)$, with $\mathfrak{S}_g$ the Siegel upper half plane of symmetric $g \times g$ complex matrices with positive definite imaginary part and

$$\Gamma_g(2n, 4n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(2n) : \operatorname{diag}(B) \equiv \operatorname{diag}(C) \equiv 0 \bmod 4n \right\},$$

where $\Gamma_g(n)$ consists of the matrices which are $\equiv I \bmod n$.

Pulling back the line bundle $\mathscr{F}_{2n}$ to $\mathfrak{S}_g$, its global sections can be expressed as classical theta constants. These are defined by

$$\theta_{m,m'}(\tau) = \sum_{p \in \mathbb{Z}^g} \exp(\pi i({}^t(p+m)\,\tau(p+m) + 2{}^t(p+m)m')).$$

Global sections of $\mathscr{F}_{2n}$ are given by the $\theta_{m,0}(2n\tau)$, with $m \in (1/2n)\,\mathbb{Z}^g/\mathbb{Z}^g$. In the case $n = 1$ these sections of $\mathscr{F}_2$ are linearly independent and we denote them by

$$\Theta_m(\tau) = \theta_{m,0}(2\tau), \qquad m \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g.$$

The space of global sections of $\mathscr{F}_4$ has two natural bases: The "canonical" basis, given by

$$\{\theta_{m,0}(4\tau) + \theta_{-m,0}(4\tau) \mid m \in \tfrac{1}{4}\,\mathbb{Z}^g/\mathbb{Z}^g\}$$

and the "classical" basis, given by

$$\{\theta_{m,m'}(\tau) \mid m, m' \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \text{ with } {}^t mm' \equiv 0 \bmod \mathbb{Z}\}.$$

We denote these by $\theta_{\mathbf{m}}$, $\mathbf{m} = (m, m')$. The relation between these bases is easily found from the Fourier series.

LEMMA 1.1. (i) *Let* $[2]$ *denote multiplication by* 2 *on the abelian variety* A. *Then* $[2]^*L^{2n} = L^{8n}$ *and the matrix with respect to the canonical bases of the linear map* $[2]^* : H^0(A, L^{2n}) \to H^0(A, L^{8n})$ *has entries which do not depend on the moduli of* A

(ii) *The matrix coefficients of the canonical map* $\operatorname{Sym}^2 H^0(A, L^{2n}) \to H^0(A, L^{4n})$ *are linear combinations of the theta constants* $\vartheta_{\beta,4n}(0)$ *of level* $4n$.

Proof. See the theta relations in [Mu2] or [I2].

For $n \mid m$, consider the canonical projection map

$$\pi_{2m,2n} : \mathscr{A}_g(2m, 4m) \to \mathscr{A}_g(2n, 4n)$$

COROLLARY 1.2. (i) *For all n, $\pi^*_{8n,2n}\mathscr{F}_{2n} \simeq \mathscr{F}_{8n}$.*

(ii) *For all n, $\pi^*_{4n,2n}\mathscr{F}^{\otimes 2}_{2n} \simeq \mathscr{F}^{\otimes 2}_{4n}$, so $\pi^*_{4n,2n}\mathscr{F}_{2n} \simeq \mathscr{F}_{4n} \otimes \mathscr{L}$ with $\mathscr{L}^{\otimes 2} \simeq \mathcal{O}$.*

Proof. (i) From the frist part of 1.1 we have a relation

$$\vartheta_{\beta,2n}(2a) = \sum_{\rho \in (\mathbb{Z}/4n)^g} c_\rho \vartheta_{\rho,8n}(a)$$

with $a \in A$. Since the c_ρ's do not depend on A, we get after putting $a = 0$, a linear relation between the theta constants, which is valid for any A. Thus both sides of the equation are sections of the same bundle, which proves (i).

(ii) is proved similarly.

Next we consider certain normal subgroups of $\Gamma(2n, 4n)$.

LEMMA 1.3. *The set of matrices*

$$I + 2n\begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$$

with B, C symmetric defines a set of coset representatives of $\Gamma(2n)/\Gamma(4n)$.

Proof. The reduction map $Sp(\mathbb{Z}) \to Sp(\mathbb{Z}/4n)$ is surjective and the order of $\Gamma(2n)/\Gamma(4n)$ is $2^{g(2g+1)}$. It is clear that the matrices above are in $\Gamma(2n)$ and that modulo $\Gamma(4n)$ there are $2^{g(2g+1)}$ of them.

Recall that $Sp(\mathbb{Z})$ is generated by matrices of the following three types

$$\text{(i)}\ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \qquad \text{(ii)}\ \begin{pmatrix} Y & 0 \\ 0 & {}^tY^{-1} \end{pmatrix}, \qquad \text{(iii)}\ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where X is a symmetric matrix.

DEFINITION. We define a map $s : \Gamma(2n)/\Gamma(4n) \to \mathbb{Z}/2$ by

$$s(r) = Tr(A + BC) \bmod 2,$$

where r is represented as in Lemma 1.3.

LEMMA 1.4. (i) *For all $u \in Sp(\mathbb{Z})$ and all $r \in \Gamma(2n)/\Gamma(4n)$ we have*

$$s(uru^{-1}) = s(r).$$

(ii) *For $r \in \Gamma(2n, 4n)/\Gamma(4n)$ we have*

$$s(r) = \mathrm{Tr}(A).$$

Proof. · (i) It suffices to prove it for the generators of $Sp(\mathbb{Z})$. For u a generator of type (i) we get

$$s(uru^{-1}) = (A + XC) + (AX + XCX + X^tA + B)C$$

For any matrix M, $\mathrm{Tr}(M^2) \equiv \mathrm{Tr}(M) \bmod 2$, so $\mathrm{Tr}((XC)^2) \equiv \mathrm{Tr}(XC) \bmod 2$. The diagonal coefficients of the symmetric matrix $AX + X^tA$ are 0 mod 2 and C is symmetric. Thus $\mathrm{Tr}((AX + X^tA)C) \equiv 0 \bmod 2$. For the other generators the argument is straightfoward.

For (ii) note that B, C are symmetric and that their diagonal entries are 0 mod 2.

PROPOSITION 1.5. *Let H be a normal subgroup of $Sp_{2g}(\mathbb{Z})$ such that $\Gamma(4n) \subset H \subset \Gamma(2n, 4n)$.*

If $H \neq \Gamma(2n, 4n)$ and $\Gamma(4n)$ then H is one of the following

(i) *H is the subgroup of index 2 in $\Gamma(2n, 4n)$ which maps onto the set of representatives of $\Gamma(2n)/\Gamma(4n)$ with*

$$\mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \bmod 2 \qquad \textit{and} \qquad \mathrm{Tr}(A) \equiv 0 \bmod 2.$$

(ii) *$\Gamma(4n)$ has index 2 in H and H maps onto the set of representatives with $B \equiv C \equiv 0$ and $A \equiv I$ or 0 mod 2.*

Proof. If H is such a normal subgroup then $H/\Gamma(4n)$ is a $Sp_{2g}(\mathbb{Z})$-stable $\mathbb{Z}/2$-subspace of $\Gamma(2n)/\Gamma(4n)$ where $Sp(\mathbb{Z})/\Gamma(2) = Sp(\mathbb{Z}/2)$ acts by conjugation. Let U be the Borel subgroup of $Sp(\mathbb{Z}/2)$ generated by the matrices above of type (i) and (ii) with Y an upper triangular matrix. Then U has an eigenvector in every stable subspace.

To find these eigenvectors in $\Gamma(2n, 4n)/\Gamma(4n)$ we first conjugate a general v, of the form given in Lemma 1.3, with a generator u of type (i). Checking $v = uvu^{-1}$ we find, that $C = 0$ and then that A must be a diagonal matrix. Taking X to be an general symmetric matrix we get $A = 0$ or $A = I$. Conjugating with a $u \in U$ of type (ii) then shows that $B_{ij} = 0$ for all i, j with $i + j > 3$. Since $v \in \Gamma(2n, 4n)/\Gamma(4n)$ we have $B_{11} = 0$. Thus any stable subspace contains at least one of the following elements:

(i) e: The representative with $A = I$, B, $C = 0$

(ii) f: The representative with $A = C = 0$, $B_{12} = B_{21} = 1$, $B_{ij} = 0$ otherwise

(iii) $e + f$

Let H be the subgroup of $\Gamma(2n, 4n)$ containing $\Gamma(4n)$ such that $H/\Gamma(4n) = \langle e \rangle$. Since e is $Sp(\mathbb{Z})$-invariant, H is normal and $[H : \Gamma(4n)] = 2$.

Next we consider the subspace W spanned by the $Sp(\mathbb{Z})$-conjugates of f. Conjugating f with a generator of type (ii) shows that W contains all reprosentatives with $A = C = 0$ and B any symmetric matrix with diagonal entries equal to 0. Conjugating with an element of type (iii) we also get all symmetric C's with diagonal entries equal to 0. Next we conjugate f with matrices which are the transpose of the type (i) generators. We then get in the upper left block, all matrices A such that the sum of the diagonal entries is 0. As $s(f) = 0$ and $f \in \Gamma(2n, 4n)$, any conjugate of f has $\mathrm{Tr}(A) = 0$ and thus every element in W has this property. If follows that W has codimension 1 in $\Gamma(2n, 4n)/\Gamma(4n)$.

If g is even then $e \in W$. The module W is then reducible but indecomposable. The following are exact sequences of $Sp(\mathbb{Z})$-modules

$$0 \to W \to \Gamma(2n, 4n)/\Gamma(4n) \to \mathbb{Z}/2 \to 0$$

$$0 \to \langle e \rangle \to W \to W' \to 0,$$

where W' is irreducible. Thus $W = \langle Sp(\mathbb{Z})(e+f) \rangle$. Hence we find only one other normal subgroup, namely the inverse image of W, in this case.

In case g is odd $\Gamma(2n, 4n)/\Gamma(4n) = \langle e \rangle \oplus W$ and W is irreducible so the $Sp(\mathbb{Z})$-module generated by $e+f$ must be all of $\Gamma(2n, 4n)/\Gamma(4n)$. Hence the only other normal subgroup H is the inverse image of W.

Remark. The $Sp_{2g}(\mathbb{Z}/2)$-module $\Gamma(2n, 4n)/\Gamma(4n)$ is in fact isomorphic to $\bigwedge^2 V$ where V is the standard representation of $Sp_{2g}(\mathbb{Z}/2)$. The isomorphism is given by

$$r \mapsto A_{ij}\, e_i \wedge e_{i+g} + B_{ij}\, e_i \wedge e_j + C_{ij}\, e_{i+g} \wedge e_{j+g},$$

where r is in the form above and $\{e_i, e_{i+g}\}$ is a symplectic basis.

Definition. Let $\Gamma(n, 2n, 4n)$ denote the unique normal subgroup of $Sp_{2g}(\mathbb{Z})$ with index 2 in $\Gamma(2n, 4n)$ and containing $\Gamma(4n)$. Thus $\Gamma(n, 2n, 4n)$ consists of the matrices $I + 2n\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\Gamma(2n)$ such that

(i) $\mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \bmod 2$

(ii) $\mathrm{Tr}(A) \equiv 0 \bmod 2$

Remarks. 1. $\Gamma(2, 4) = \Gamma(1, 2, 4)$ if and only if g is odd. Indeed for any g, $-I$ is in $\Gamma(2, 4)$ but not in $\Gamma(4)$. Since $-I_{2g} = I_{2g} + 2\,\mathrm{diag}(-I_g, -I_g)$ we have $-I_{2g} \in \Gamma_g(1, 2, 4)$ if and only if $\mathrm{Tr}(-I_{2g}) \equiv 0 \bmod 2$, i.e., if and only if g is even.

2. *If* $g = 1$ one has $\Gamma(n, 2n, 4n) = \Gamma(4n)$ for every n. This follows from a computation of the indices of these groups in $SL_2(\mathbb{Z})$.

THEOREM 1.6. (i) *The line bundles* $\mathcal{F}_{4n}$ *and* $\pi^*_{4n,2n}\mathcal{F}_{2n}$ *are isomorphic if and only if n is even.*

(ii) *Assume n is odd. Then there exists a non-trivial line bundle* $\mathcal{L}$ *on* $\mathcal{A}_g(4n, 8n)$ *with* $\mathcal{L}^2 \simeq \mathcal{O}$ *such that* $\pi^*_{4n,2n}\mathcal{F}_{2n} \simeq \mathcal{F}_{4n} \otimes \mathcal{L}$.

(ii) *Assume n is odd. Let* $\mathcal{A}_g(2n, 4n, 8n)$ *denote the unramified* $2:1$ *cover of* $\mathcal{A}_g(4n, 8n)$ *defined by* $\mathcal{L}$. *Then over* $\mathbb{C}$, $\mathcal{A}_g(2n, 4n, 8n) \simeq \mathfrak{S}_g/\Gamma(2n, 4n, 8n)$ *so the map from* $\mathfrak{S}_g$ *to projective space defined by the global sections of either* $\mathcal{F}_{2n}$ *or* $\mathcal{F}_{4n}$ *factors over* $\mathcal{A}_h(2n, 4n, 8n)$ *but not over* $\mathcal{A}_g(4n, 8n)$.

Proof. We first give the proof in the case where the ground field is $\mathbb{C}$. By Corollary 1.2(ii) we have $\pi^*_{4n,2n}\mathcal{F}_{2n} \simeq \mathcal{F}_{4n} \otimes \mathcal{L}$ with $\mathcal{L}^2 \simeq \mathcal{O}$. Put $F_n(\tau) = \theta_{00}(2n\tau)/\theta_{00}(4n\tau)$ and $Q(M) = F_n(M\tau)/F_n(\tau)$ for $M \in \Gamma(4n, 8n)$.

Since $\theta_{00}(2n\tau)$ defines a global section of $\pi^*_{4n,2n}\mathcal{F}_{2n}$ and $\theta(4n\tau)$ a global section of $\mathcal{F}_{4n}$ we have $\mathcal{L} \simeq \mathcal{O}$ if and only if $Q(M) = 1$ for all $M \in \Gamma(4n, 8n)$. Now $\mathcal{L}^2 \simeq \mathcal{O}$ so $Q : \Gamma(4n, 8n) \to \{\pm 1\}$ and hence ker Q is a normal subgroup of index at most 2. It is easy to see that ker Q contains the matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ with $X_{11} = 8n$ and all the other entries 0. In case $g \leqslant 2$ it follows from [Me] that $\Gamma(8n) \subset \ker Q$. By Proposition 1.5 it then follows that ker Q contains $\Gamma(2n, 4n, 8n)$. The group $\Gamma(4n, 8n)/\Gamma(2n, 4n, 8n) \simeq \mathbb{Z}/2$ is generated by the matrix

$$M_g = \begin{pmatrix} 1+4n & 0 & 8n^2 & 0 \\ 0 & I_{g-1} & 0 & 0 \\ 8n & 0 & 1-4n+16n^2 & 0 \\ 0 & 0 & 0 & I_{g-1} \end{pmatrix}.$$

Note that $2nM_g(\tau) = 2M'_g(n\tau)$ where

$$M'_g = \begin{pmatrix} 1+4n & 0 & 8n^3 & 0 \\ 0 & 1 & 0 & 0 \\ 8 & 0 & 1-4n+16n^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Writing τ' for $n\tau$ (so $F_n(\tau) = F_1(\tau')$) an choosing for τ a period matrix with $\tau_{1i} = 0$ for $i > 1$, we get $Q(M) = \theta_{00}(2M'_1\tau'_{11})\,\theta_{00}(4\tau'_{11})/\theta(4M'_1\tau'_{11})\,\theta_{00}(2\tau'_{11})$.

From the transformation formulas for the theta functions $\theta_{00}(\tau)$ and $\theta_{00}(2\tau)$ (cf. [Ko]) it follows that $Q(M_g) = 1$ if and only if $M'_1 \in \Gamma_1(8) = \Gamma_1(2, 4, 8)$. Since $M'_1 \in \Gamma(8)$ if and only if n is even it follows that Q is trivial if and only if n is even.

In the general case of a field k of characteristic $\nmid 2n$ the result follows from the fact that the situation over $\mathbb{C}$ is obtained by base change from

$\mathbb{Z}[1/2n]$ and that the line bundle of order 2 can not have trivial reduction at any prime of $\mathbb{Z}[1/2n]$. This concludes the proof of 1.6.

From now on we assume $g=2$. We denote the Satake compactification of $\mathscr{A}(*)$ by $\mathscr{A}^s(*)$.

The following proposition follows from Igusa's results on modular forms of genus 2.

PROPOSITION 1.7. *The morphism*

$$\Theta_2 : \mathscr{A}(2,4) \to \mathbb{P}^3$$

extends to an isomorphism

$$\mathscr{A}^s(2,4) \to \mathbb{P}^3.$$

The closure of the locus of those ppav's which are products of two elliptic curves (as ppav's) consists of 10 *quadrics in* $\mathbb{P}^3$. *The boundary of* $\mathscr{A}^s(2,4)$ *consists of* 30 *lines. These lines are the intersections of pairs of the quadrics. Any pair of the quadrics intersect in* 4 *lines configured in a square. Under the map* $\mathscr{A}^s(2,4) \to \mathscr{A}^s(2)$ *the two opposing sides in a square map to the same boundary component.*

Furthermore the morphism

$$\Theta_4 : \mathscr{A}(4,8) \to \mathbb{P}^9$$

extends to an embedding

$$\mathscr{A}^s(4,8) \to \mathbb{P}^9.$$

Proof. The Satake compactification is the projective variety $\mathscr{A}^s(2,4) =$ Proj $R(\Gamma(2,4))$ where $R(\Gamma(2,4))$ is the graded ring of modular forms on $\Gamma(2,4)$. We have $R(\Gamma(2,4)) = \mathbb{C}[(\theta_{\mathbf{m}}\theta_{\mathbf{n}})^2, \theta]$ ([I2, p. 397]) where θ is the product of all the 10 non-vanishing theta constants (this is a cusp form of weight 5 for $\Gamma(2)$). Note that the other generators have weight 2 and that θ^2 is in the subring generated by the other generators. Then we have $\operatorname{Proj} R(\Gamma(2,4)) \simeq \operatorname{Proj} R(\Gamma(2,4))_{\text{even}} \simeq \operatorname{Proj} \mathbb{C}[(\theta_{\mathbf{m}}\theta_{\mathbf{n}})^2] \simeq \operatorname{Proj} \mathbb{C}[\theta_{\mathbf{m}}^2]$. The formulas in [I2, p. 408] imply that $\mathbb{C}[\theta_{\mathbf{m}}^2]$ is the subring of $\mathbb{C}[\Theta_m]$ generated by the elements of degree 2. Hence $\operatorname{Proj} \mathbb{C}[\theta_{\mathbf{m}}^2] = \operatorname{Proj} \mathbb{C}[\Theta_m]$. Since $\mathscr{A}^s(2,4)$ has dimension 3 and $\mathbb{C}[\Theta_m]$ is generated by the $4\Theta_m$'s, there are no relations among the Θ_m's so $\mathbb{C}[\Theta_m]$ is the polynomial ring in 4 variables hence its Proj is $\mathbb{P}^3$.

A 2-dimensional ppav is a product of two elliptic curves of and only if its theta divisor is reducible. This is equivalent to the vanishing of exactly one of the 10 theta constants. The theta relations below give the connections with the quadrics. Since any boundary point is the limit of a family of products of elliptic curves, each boundary component lies in one of the quadrics.

The 9 remaining theta constants on a product of two elliptic curves are the 9 products of the 3 elliptic modular theta constants on each of the elliptic curves. On the boundary where one or both of the elliptic curves degenerate to $\mathbb{G}_m$ at least one of these theta constants must vanish and hence at least 3 of the 9 products vanish. It follows that the boundary components are characterized by the vanishing of at least 4 theta constants.

By a direct verification one finds that the union of the intersections of all pairs of the 10 quadrics of 30 lines and each line lies on 4 of the quadrics.

The last statement follows from the fact that

$$\mathscr{A}^s(4,8) \simeq \operatorname{Proj} R(\Gamma(4,8)) \simeq \operatorname{Proj} \mathbb{C}[\theta_{\mathbf{m}}\theta_{\mathbf{n}}] \simeq \operatorname{Proj} \mathbb{C}[\theta_{\mathbf{m}}].$$

Next we consider the morphism

$$\Theta_{(2,4,8)} : \mathfrak{S}_2 \to \mathbb{P}^{13}$$

defined by

$$\Theta_{(2,4,8)}(\tau) = (\cdots : \theta_{\mathbf{m}}(\tau) : \cdots : \Theta_m(\tau) : \cdots),$$

i.e., defined by the 10 $\theta_{\mathbf{m}}$ and the 4 $\Theta_m(\tau)$.

The Riemann theta relations below ([I1]) imply that the image is contained in the intersection of 10 quadrics:

$$\begin{aligned}
\theta_{0000}^2 &= \Theta_{00}^2 + \Theta_{01}^2 + \Theta_{10}^2 + \Theta_{11}^2 \\
\theta_{0001}^2 &= \Theta_{00}^2 - \Theta_{01}^2 + \Theta_{10}^2 - \Theta_{11}^2 \\
\theta_{0010}^2 &= \Theta_{00}^2 + \Theta_{01}^2 - \Theta_{10}^2 - \Theta_{11}^2 \\
\theta_{0011}^2 &= \Theta_{00}^2 - \Theta_{01}^2 - \Theta_{10}^2 + \Theta_{11}^2 \\
\theta_{0100}^2 &= 2(\Theta_{000}\Theta_{01} + \Theta_{10}\Theta_{11}) \\
\theta_{1000}^2 &= 2(\Theta_{00}\Theta_{10} + \Theta_{01}\Theta_{11}) \\
\theta_{1100}^2 &= 2(\Theta_{00}\Theta_{11} + \Theta_{01}\Theta_{10}) \\
\theta_{0110}^2 &= 2(\Theta_{00}\Theta_{01} - \Theta_{10}\Theta_{11}) \\
\theta_{1001}^2 &= 2(\Theta_{00}\Theta_{10} - \Theta_{01}\Theta_{11}) \\
\theta_{1111}^2 &= 2(\Theta_{00}\Theta_{11} - \Theta_{01}\Theta_{10}).
\end{aligned}$$

THEOREM 1.8. (i) *The map* $\Theta_{(2,4,8)}$ *defines an embedding of* $\mathscr{A}(2,4,8) := \mathfrak{S}_2/\Gamma(2,4,8)$ *into* $\mathbb{P}^{13}$. *The closure of the image is the complete intersection of the* 10 *quadrics above.*

(ii) *The projection on the last* 4 *coordinates* (*the* Θ_m*'s*) *induces the canonical map* $\mathscr{A}^s(2,4,8) \to \mathscr{A}^s(2,4) = \mathbb{P}^3$. *The covering group is* $(\mathbb{Z}/2)^{10}$ *acting by changing the signs of the* 10 $\theta_{\mathbf{m}}$*'s.*

(iii) *The projection on the first* 10 *coordinates induces the canonical* $2:1$ *map* $\mathscr{A}^s(2,4,8) \to \mathscr{A}^s(4,8)$. *The covering involution is induced by changing simultaneously the sign of the last* 4 *coordinates. This involution acts fixed point free.*

Proof. We have already shown that $\Theta_{(2,4,8)}$ factors over $\mathfrak{S}_2/\Gamma(2,4,8)$ which is a $2:1$ cover of $\mathscr{A}(4,8)$. The quadratic relations above imply that its image is a $2:1$ cover of the image of $\mathscr{A}(4,8)$ under Θ_4. Thus (i) follows. The others are straightforward.

Consider now a 6-tuple of distinct even theta characteristics and consider the remaining 4. The corresponding 4 theta constants together with the 4 Θ's define a map $\mathscr{A}(2,4,8) \to \mathbb{P}^7$ which is of course just the projection on the appropriate coordinates from the image of $\Theta_{(2,4,8)}$. The closure of the image is the complete intersection defined by the 4 corresponding theta relations. Let Γ' be the stabilizer of this map in $\Gamma(2,4)$ then we get an embedding $\mathfrak{S}_2/\Gamma'$ into the complete intersection in $\mathbb{P}^7$. Since the product of all 10 theta characteristics is a modular form for $\Gamma(2)$, the product of the 6 theta constants is a modular form of weight 6 for Γ'. If this form is a cusp form it defines a holomorphic 3-form on every resolution of the complete intersection.

In the next section we study these complete intersections.

2. Siegel Modular Threefolds Associated to Products of Theta Constants

The set of 6-fold products of distinct theta constants contains $\binom{10}{6} = 210$ elements. These are all modular forms of weight 3 for $\Gamma(4,8)$.

Under the action of $Sp_4(\mathbb{Z})$ this set breaks into 3 orbits:

(1) An orbit consisting of 15 forms, characterized by the sum of the theta characteristics being 0 in $(\mathbb{Z}/2)^4$. These are all cusp forms, in fact they are precisely the cusp forms of weight 3 for the principal congruence subgroup $\Gamma(4)$.

(2) An orbit consisting of 180 forms, characterized by the set of 6 theta characteristics being the union of an asyguous and a syzyguous triple. These are all cusp forms.

(3) Finally the remaining 15 forms from an orbit. These are not cusp forms.

The action of $Sp_4(\mathbb{Z})$ on $\mathcal{A}(2, 4, 8)$ is defined over $\mathbb{Q}(\zeta_8)$ and hence the complete intersections associated to the forms in an orbit are all isomorphic to each other over $\mathbb{Q}(\zeta_8)$. Thus it suffices to consider one form from each orbit.

As a representative for the orbit 1) we take the 6-tuple of theta characteristics

$$(0100, 0110, 1000, 1001, 1100, 1111).$$

The remaining 4 theta constants and the Θ_m's define a map

$$(\theta_{0000} : \theta_{0001} : \theta_{0010} : \theta_{0011} : \Theta_{00} : \Theta_{01} : \Theta_{10} : \Theta_{11}) : \mathfrak{S}_2 \to \mathbb{P}^7$$

The theta relations show that the closure of the image is the complete intersection $X \subset \mathbb{P}^7$ defined by the equations

$$Y_0^2 = X_0^2 + X_1^2 + X_2^2 + X_3^2$$
$$Y_1^2 = X_0^2 - X_1^2 + X_2^2 + X_3^2$$
$$Y_2^2 = X_0^2 + X_1^2 - X_2^2 - X_3^2$$
$$Y_3^2 = X_0^2 - X_1^2 - X_2^2 + X_3^2.$$

PROPOSITION 2.1. (i) *The singular locus of X consists of* 96 *ordinary double points.*

(ii) *The Euler characteristic of X is* -32.

(iii) *Let X' be the blow-up of X along its singular locus. Then the Euler number of X' is* 256.

Proof. A singular point must have at least 2 of its Y coordinates equal to 0. One finds easily that there are 96 singular points all rational over $\mathbb{Q}(\zeta_8)$ and that they are all ordinary double points.

The Euler characteristic can be determined from the vanishing cycle exact sequence

$$0 \to H^3(X, \mathbb{Q}) \to H^3(X_\eta, \mathbb{Q}) \to V \to H^4(X, \mathbb{Q}) \to H^4(X_\eta, \mathbb{Q}) \to 0$$
$$H^i(X, \mathbb{Q}) = H^i(X_\eta, \mathbb{Q}) \qquad \text{for} \quad i \neq 3, 4.$$

Here V is the space of vanishing cycles, in this case the $\mathbb{Q}$-vectorspace spanned by the tangent cones, so dim $V = 96$, and X_η is a general complete intersection of 4 quadrics in $\mathbb{P}^7$. Thus $\chi(X) = \chi(X_\eta) + 96$. Since $\chi(X_\eta) = -128$, (ii) follows.

The Leray spectral sequence for the blow-up of the double points $X' \to X$ gives exact sequences of terms of low degree

$$0 \to H^2(X, \mathbb{Q}) \to H^2(X', \mathbb{Q}) \to H^2(E, \mathbb{Q}) \to H^3(X, \mathbb{Q}) \to H^3(X', \mathbb{Q}) \to 0$$
$$0 \to H^4(X, \mathbb{Q}) \to H^4(X', \mathbb{Q}) \to H^4(E, \mathbb{Q}) \to 0$$
$$H^i(X, \mathbb{Q}) = H^i(X', \mathbb{Q}) \qquad \text{for} \quad i \neq 2, 3, 4.$$

Here E is the exceptional fiber, i.e., the union of 96 quadrics in $\mathbb{P}^3$. It follows that $\chi(X') = \chi(X) + 288 = 256$.

PROPOSITION 2.2. *The Betti and Hodge numbers of X' are*

$$\begin{array}{ll} h^0 = h^6 = 1 & \\ h^1 = h^5 = 0 & h^{2,0} = 0, h^{1,1} = 128 \\ h^2 = h^4 = 128 & h^{3,0} = 1, h^{2,1} = 0 \\ h^3 = 2 & \end{array}$$

Proof. From the Leray spectral sequence for the blow-up one finds that $h^1 = h^5 = 0$, and that H^2 is spanned by divisors so $h^{2,0} = 0$. The Kaehler differential coming from $\mathbb{P}^7$ via Poincare residue pulls-back to a regular 3-form on X', hence $h^{3,0} = 1$. Using this and the computation of the Euler characteristic we get

$$h^2 - h^3 + h^4 = 2h^2 - h^3 = 254$$

The number of points on X defined over $\mathbb{F}_{17}$ is 13024, counted by a computer. Since $17 \equiv 1 \bmod 8$, all the double points as well as the rulings on the exceptional fibers are rational. This adds $96(2p + p^2)$, $p = 17$ points. Hence we have $\# X'(\mathbb{F}_{17}) = 44032$.

On the other hand $\# X'(\mathbb{F}_p) = t_0(p) + t_2(p) - t_3(p) + t_4(p) + t_6(p) = 1 + (1+p)h^2 - t_3(p) + p^3$ where $t_i(p) = \mathrm{Tr}(\mathrm{Frob}_p : H^i(X', \mathbb{Q}_l))$. Here we have used that H^2 is spanned by classes of divisors so Frobenius acts by multiplication by p.

Since $|t_3(p)| \leqslant p^{3/2} h^3$ we have

$$|(1 + p(1+p)h^2 - \# X'(\mathbb{F}_{17})| \leqslant (2h^2 - 254)\, p^{3/2}$$

It follows from this estimate that $h^2 = 128$ and so $h^3 = 2$ and $t_3 = 50$.

Next we are going to explicitly determine the semi-simplification of the Galois representation

$$\rho : G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(H^3(X'_{\bar{\mathbb{Q}}}, \mathbb{Q}_l))$$

We need to know the trace $t_3(p)$ for several primes:

LEMMA 2.3. *Let N_p denote the number of points on X over the field $\mathbb{F}_p$. Then*

$$t_3(p) = (1 - 64p + 32p^2 + p^3) - N_p, \qquad p \equiv 1 \bmod 8$$

$$t_3(p) = (1 + p^3) - N_p, \qquad p \equiv 3, 5 \bmod 8$$

$$t_3(p) = (1 - 16p + 8p^2 + p^3) - N_p, \qquad p \equiv 7 \bmod 8$$

Proof. The first case was proved in the preceding proposition. In case $p \equiv 3, 5 \bmod 8$, none of the singular points are rational. Hence $t_3(p) = 1 + p(1+p)k + p^3 - N_p$ where k is the difference between the multiplicities of the eigenvalues $+p$ and $-p$ (remark that all of H_l^2 is algebraic over $\mathbb{F}_{p^2}$ so $\mathrm{Frob}_p^2 = p^2$). Since all the double points and the rulings on the exceptional fibers in the blow-up are rational over $\mathbb{Q}(\zeta_8)$, k depends only on p mod 8. To determine k we use the estimate on $|t_3(p)|$ as before. Computing $N_{19} = 6818$ and $N_{29} = 24192$ we find $k = 0$ in both cases and then $t_3(19) = 44$, $t_3(29) = 198$.

In case $p \equiv 7 \bmod 8$, 24 of the double points are rational over $\mathbb{F}_p$ and the rulings on the exceptional fibers are also rational. It follows that $\# X'(\mathbb{F}_p) = N_p - 24(2p + p^2)$, and $t_3(p) = 1 + p(1+p)k + p^3 - \# X'(\mathbb{F}_p)$. Computing $N_{23} = 16088$ one finds $k = 32$ and $t_3(23) = -56$.

Let

$$g = q \prod (1 - q^{2n})^4 (1 - q^{4n})^4 = \eta(2z)^4 \, \eta(4z)^4, \qquad q = e^{2\pi i z}$$

Then g is the unique new-form of weight 4 for $\Gamma_0(8)$.

THEOREM 2.4. *We have an identity of L-functions*

$$L(\rho, s) = L(g, s)$$

Proof. Consider the Galois representation r_g associated to the new-form g [D]. We may assume $r_g : G \to GL(2, \mathbb{Z}_2)$ and $L(g, s) = L(r_g, s)$. Also we can consider ρ as a representation in $GL(2, \mathbb{Z}_2)$. To show that the semi-simplifications of these two representations are isomorphic we follow the method in [L1].

The determinants of ρ and r_g are the same. Both representations are unramified outside 2 and if we reduce mod 2 we get representations in $GL(2, \mathbb{F}_2) = S_3$. The image of the two representatons is either contained in $\mathbb{Z}/2$ or is all of S_3. But a surjective map $G \to S_3$, ramified only at 2, would give rise to a quadratic extension of $\mathbb{Q}$, unramified outside 2, which has a cubic extension unramified outside the prime lying over 2. Now the quadratic extensions of $\mathbb{Q}$ ramified only at 2 are $\mathbb{Q}[i]$, $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{-2}]$ but none of these has a cubic extension unramified outside the prime over 2. It follows that $\operatorname{Tr}\rho(\sigma) \equiv \operatorname{Tr} r_g(\sigma) \equiv 0 \bmod 2$ for all $\sigma \in G$. The compositum of all the quadratic extensions of $\mathbb{Q}$ unramified outside 2 is the field $\mathbb{Q}[i, \sqrt{2}]$. The non-trivial elements in $\operatorname{Gal}(\mathbb{Q}[i, \sqrt{2}]/\mathbb{Q})$ are the frobenius elements at 3, 5, 7 so they certainly form a non-cubic set. Computing the traces $\operatorname{Tr}\rho(\operatorname{frob}_p) = t_3(p)$ using Lemma 2.3 we get that $\operatorname{Tr}\rho(\operatorname{frob}_p) = \operatorname{Tr} r_g(\operatorname{frob}_p)$ for these primes. By [Li, Thm. 4.2], it follows that the semi-simplifications are isomorphic.

Remarks. The threefold X has also been studied, independently and for different reasons, by Hirzebruch [Hi]. Indeed, composing the map $X \to \mathscr{A}^S(2, 4) = \mathbb{P}^3$ with the map $\mathbb{P}^3 \to \mathbb{P}^3$ which squares the coordinates, we realize X as a covering of $\mathbb{P}^3$ with group $(\mathbb{Z}/2)^7$, branched over the union of the 4 coordinate hyperplanes and the 4 hyperplanes defined by the equations

$$X_0 + X_1 + X_2 + X_3 = 0$$

$$X_0 - X_1 + X_2 - X_3 = 0$$

$$X_0 + X_1 - X_2 - X_3 = 0$$

$$X_0 - X_1 - X_1 + X_3 = 0.$$

It is easy to verify that these hyperplanes form the faces of a regular octahedron.

Let $\mathscr{E}$ denote the universal elliptic curve over the modular curve $X_0(8)$ and let W denote the resolution of the fiber product $\mathscr{E} \times_{X_0(8)} \mathscr{E}$. Then W is a threefold and the cusp form $g \in S_4(\Gamma_0(8))$ gives rise to a holomorphic 3-form on W. The Galois representation r_g occurs in $H^3(W, \mathbb{Q}_l)$ and hence the trivial representation occurs in $H^6(W \times X', \mathbb{Q}_l(3))$. The Tate conjecture predicts the existence of a correspondence between W and X' defined over $\mathbb{Q}$, inducing an isomorphism between the Galois representations on the H^3's. Such a correspondence has been constructed by J. Stienstra (unpublished) (see [E-vG] for a moduli theoretic construction of a corresponce). We shall briefly explain Stienstra's construction: By Beauville ([Be]), the universal curve $\mathscr{E}$ over $X_0(8) = \mathbb{P}^1$ can be realized as the family

$$tXYZ + s(X - Y)(Z^2 - XY) = 0,$$

$$(s : t) \in X_0(8), (X : Y : Z) \in \mathbb{P}^2.$$

The map $\mathscr{E} \to X_0(8)$ is given by $(t, (X: Y: Z)) \mapsto t$ so on the affine open piece $s = 1$, $Z = 1$ we have

$$t = \frac{(X-Y)(-1+XY)}{XY} = X + \frac{1}{X} - Y - \frac{1}{Y}.$$

The fiber product $\mathscr{E} \times_{X_0(8)} \mathscr{E}$ is then given by the affine equation

$$X + \frac{1}{X} - Y - \frac{1}{Y} = U + \frac{1}{U} - V - \frac{1}{V}.$$

Changing variables we obtain an affine equation for W:

$$U_0 + U_0^{-1} + U_1 + U_1^{-1} + U_2 + U_2^{-1} + U_3 + U_3^{-1} = 0.$$

To construct a rational map $X \to W$ we consider the intermediate threefold V defined by the equations

$$\begin{aligned} V_0^2 &= W_0(W_0 - W_1 - W_2 - W_3) \\ V_1^2 &= W_1(-W_0 + W_1 - W_2 - W_3) \\ V_2^2 &= W_2(-W_0 + W_1 + W_2 - W_3) \\ V_3^2 &= W_3(-W_0 - W_1 - W_2 + W_3). \end{aligned}$$

This is the 16-fold cover of $\mathbb{P}^3$ ramified over the 4 pairs of opposite sides in the regular octahedron considered in Hirzebruch's construction.

It is now straightforward to verify that the following are well-defined dominant rational maps

$$\phi: X \to V,\ V_i := X_i Y_i,\ 0 \leqslant i \leqslant 3, \qquad W_0 := -X_0^2,\ W_i := X_i^2,\ 1 \leqslant i \leqslant 3$$

$$\psi: V \to W,\ U_i := -1 + 2\sqrt{2}\,\frac{V_i + W_i\sqrt{2}}{W_0 + W_1 + W_2 + W_3},\ 0 \leqslant i \leqslant 3.$$

For the sake of completeness we prove the following proposition.

PROPOSITION 2.5. *Let Γ' be the congruence subgroup defining X. Then X is the Satake compactification of $\mathfrak{S}_2/\Gamma'$.*

Proof. Composing the map $X \to \mathbb{P}^7$ with some $2n$-tuple embedding of $\mathbb{P}^7$ into $\mathbb{P}^N$, we see that X is embedded by modular forms on Γ'.

The Satake compactification $\mathscr{A}^s(\Gamma') = \operatorname{Proj} R(\Gamma')$ so we have a birational morphism $\mathscr{A}^s(\Gamma') \to X$ which is an isomorphism on an open set of $\mathfrak{S}_2/\Gamma'$. We show that this map is a bijection. This is clear on the open set $\mathfrak{S}_2/\Gamma'$ by proposition 1.6. To see that it is bijective on the boundary, note that the covering involution of $\Theta_{(2,4,8)}(\mathscr{A}^s(2,4,8))$ over $\Theta_4(\mathscr{A}^s(4,8)) \simeq \mathscr{A}^s(4,8)$ is fixed point free, thus $\Theta_{(2,4,8)}$ must induce a bijection between $\mathscr{A}^s(2,4,8)$ and its image. Since X is the quotient of this image by the finite group $\Gamma'/\Gamma(2,4,8)$ and the Satake compactification $\mathscr{A}^s(\Gamma')$ is the quotient of $\mathscr{A}^s(2,4,8)$ by the same group, it follows that the birational morphism above is bijective. Since X is normal it follows from Zariski's Main Theorem that the map is an isomorphism.

Next we consider the orbit of 6-tuples of theta characteristics which are unions of an azygyous and a syzygyous triple (a triple of theta characteristics is azygyous if the sum of the theta characteristics in the triple is an odd characteristic, syzygyous if it is an even characteristic). This orbit has 180 elements and the corresponding products of theta constants are cusp forms of weight 3 for $\Gamma(4,8)$. We take as a representative the 6-tuple

$$(0000, 0001, 0010, 0011, 1001, 1111).$$

The Riemann theta relations then give the complete intersection Y defined by the equations

$$\begin{aligned} Y_0^2 &= 2(X_0X_1 + X_2X_3) \\ Y_1^2 &= 2(X_0X_2 + X_1X_3) \\ Y_2^2 &= 2(X_0X_3 + X_1X_2) \\ Y_3^2 &= 2(X_0X_1 - X_2X_3). \end{aligned}$$

The Euler characteristic can be computed by viewing Y as a sequence of four 2 : 1 coverings

LEMMA 2.6. $\chi(Y) = -12$.

Proof. Let $Y(i) \subset \mathbb{P}^{i+3}$ be the threefold defined by the first i equations. Let $Q(i) \subset Y(i)$ be the branch locus of the covering $Y(i) \to Y(i-1)$ so $Q(i)$ is the surface defined by the first $i-1$ equations and $Y_i = 0$. We have $\chi(Y(i)) = 2\chi(Y(i-1)) - \chi(Q(i))$. Each of the surfaces $Q(i)$ maps by a sequence of $i-1$ degree 2 coverings to the quadric in $\mathbb{P}^3$ defined by the right hand side of the ith equation. The ramification locus for each of these 2 : 1 coverings can be found from the intersections of the 4 quadrics in $\mathbb{P}^3$. Computing the Euler characteristics successively we find for $i = 1, 2, 3, 4$ respectively $\chi(Q(i)) = 4, 4, 8, 12$ and $\chi(Y(i)) = 4, 4, 0, -12$.

PROPOSITION 2.7. (i) *The singular locus of Y consists of* 16 *ordinary double points and* 4 *plane conics intersecting transversally, configured in a square.*

(ii) *A resolution Y' is obtained by first blowing up the* 16 *double points and a pair of opposite sides in the square of conics and then blowing up the strict transforms of the other pair of conics.*

(iii) *The fiber of $\pi : Y' \to Y$ over a double point is a smooth quadric in $\mathbb{P}^3$. The fibers over a point P on one of the conics are as follows*:

(a) $\pi^{-1}(P) = \mathbb{P}^1$ *if P is not on any of the other conics and if P is not one of* 8 *"special ponts" lying on each of the* 4 *conics.*

(b) *If P is on the intersection of two of the conics, $\pi^{-1}(P)$ is a tree of* 3 $\mathbb{P}^1$'s.

(c) *If P is one of the special points $\pi^{-1}(P)$ is a tree of* 2 $\mathbb{P}^1$*'s.*

(iv) *The Euler characteristic of Y' is* 80.

Proof. It is easy to check that a singular point must have at least 2 Y-coordinates equal to 0. One finds 16 double points

$$(\pm\sqrt{2} : 0 : 0 : \pm\sqrt{2} : 1 : 1 : 0 : 0)$$
$$(\pm\sqrt{-2} : 0 : 0 : \pm\sqrt{-2} : 1 : -1 : 0 : 0)$$
$$(\pm\sqrt{2} : 0 : 0 : \pm\sqrt{-2} : 0 : 0 : 1 : 1)$$
$$(\pm\sqrt{-2} : 0 : 0 : \pm\sqrt{2} : 0 : 0 : 1 : -1).$$

The other singular points are on the 4 conics defined by

$$Y_0 = Y_2 = Y_3 = X_0 = X_2 = 0$$
$$Y_0 = Y_2 = Y_3 = X_1 = X_3 = 0$$
$$Y_0 = Y_1 = Y_3 = X_0 = X_3 = 0$$
$$Y_0 = Y_1 = Y_3 = X_1 = X_2 = 0.$$

To describe the resolution we consider the situation locally around a point on the intersection of two of the conics e.g., $P = (0:0:0:0:1:0:0:0)$. On the open set of $\mathbb{P}^7$ where $X_0 \neq 0$ and $(Y_0^2 + Y_3^2)^2 \neq 16$ we can put $X_0 = 1$ and eliminate the other X_i's. First we get

$$X_1 = \tfrac{1}{4}(Y_0^2 + Y_3^2)$$
$$X_2 = \tfrac{1}{2}(Y_1^2 - \tfrac{1}{2}(Y_0^2 + Y_3^2)X_3)$$
$$X_3 = \tfrac{1}{2}(Y_2^2 - \tfrac{1}{2}(Y_0^2 + Y_3^2)X_2)$$
$$X_2X_3 = \tfrac{1}{4}(Y_0^2 - Y_3^2).$$

Eliminating the X_i's we get the equation

$$UY_0^2 - VY_3^2 - WY_1^2Y_2^2 = 0, \qquad (*)$$

where

$$U = (16 - (Y_0^2 + Y_3^2)^2)^2 + 64 Y_1^2 + 64 Y_2^2$$

$$V = (16 - (Y_0^2 + Y_3^2)^2)^2 - 64 Y_1^2 - 64 Y_2^2$$

$$W = 16(16 - (Y_0^2 + Y_3^2)^2).$$

Near P the functions U, V, W are units and hence passing to an etale cover we can transform them away so the Eq. (*) simply becomes

$$Y_0^2 - Y_3^2 - Y_1^2 Y_2^2 = 0.$$

Blowing up this hypersurface, first in the line $L_1 : Y_0 = Y_2 = Y_3 = 0$ and then in the strict transform of the line $L_2 : Y_0 = Y_1 = Y_3 = 0$ we get a smooth variety. We briefly indicate how the procedure works:

For the first blow up we put $Y_0 = U_1 Y_2$, $Y_3 = U_2 Y_2$. In $\mathbb{A}^4$ with coordinates (U_1, U_2, Y_1, Y_2) we get the equation

$$U_1^2 - U_2^2 - Y_1^2 = 0.$$

The fiber over the line parametrized by $(Y_0, Y_1, Y_2, Y_3) = (0, 0, t, 0)$ is then given by

$$U_1^2 - U_2^2 - t^2 = 0$$

so for $t \neq 0$ the fiber is a smooth conic $\simeq \mathbb{P}^1$ and for $t = 0$ we get the two lines $(U_1, U_2, Y_1, Y_2) = (s, \pm s, 0, 0)$.

Next we blow up the other line, which in the new coordinates is given by $(U_1, U_2, Y_1, Y_2) = (0, 0, 0, t)$.

Taking $U_2 = V_1 U_1$, $Y_1 = V_2 U_1$ we get the equation

$$V_2^2 - V_1^2 - 1 = 0$$

in $\mathbb{A}^4$ with coordinates (U_1, V_1, V_2, Y_2). The fiber is now a smooth conic in the V_1, V_2-plane and does not depend on the point on the line. To see how the two lines we found earlier intersect the conic, we observe that over the origin in $\mathbb{A}^4$ with coordinates (Y_0, Y_1, Y_2, Y_3) we have after the two blow ups, the two planes defined by the equation $V_2 U_1 = 0$. The conic lies in the plane defined by $U_1 = 0$ and the two lines lie in the plane defined by $V_2 = 0$. They meet at the points where $V_1 = \pm 1$, one point on each line.

The special points on L_2 are by definition the points where $UV = 0$ i.e., $16^2 + 64 Y_2^4 = 64(4 + Y_2^4) = 0$ or $64(4 - Y_2^4) = 0$.

In a neighborhood of the special point Q with $Y_2 = \sqrt[4]{4}$ we may then write

$$V = (Y_2 - \sqrt[4]{4})\, V' + \text{higher order terms}$$

so Y_0, Y_1, V, Y_3 are local coordinates at Q. Since Y_2 does not vanish at Q neither does $W' = WY_2^4$ and hence after an etale extension the equation for Y may be written as

$$UY_0^2 - VY_3^2 - W'Y_1^2 = 0 \text{ or } -VY_3^2 + Y_1^2 = 0.$$

It is now clear that for $V \neq 0$ the fiber of the blow up over a point of L_2 is a smooth conic and for $V = 0$ it consists of 2 lines, intersecting in one point. The situation at the other special points is similar.

To compute the Euler characteristic of Y' we use the description above and Mayer–Vietoris sequences. The 16 double points contribute 48 and the fiber over the singular lines contribute 48 thus $\chi(Y') = -12 + 44 + 48 = 80$.

Next we determine the Betti and the Hodge numbers of Y'.

PROPOSITION 2.8. *The Betti and Hodge numbers of Y' are*

$$\begin{aligned} &h^0 = h^6 = 1 \\ &h^1 = h^5 = 0 \qquad h^{2,0} = 0, \qquad h^{1,1} = 41 \\ &h^2 = h^4 = 41 \qquad h^{3,0} = 1, \qquad h^{2,1} = 1. \\ &h^3 = 4 \end{aligned}$$

Proof. From the Leray spectral sequence for the blow up we get $h^1 = h^5 = 0$ and also that H^2 is spanned by algebraic classes so $h^{2,0} = 0$. The space of cusp forms of weight 3 is in $1-1$ correspondence with the space of holomorphic 3-forms on the toroidal compactification and since $h^{3,0}$ is a birational invariant, we get $h^{3,0} = 1$. Using this and the Euler characteristic computed above, we get $h^2 - h^3 + h^4 = 2h^2 - h^3 = 78$.

The number of points on Y over the finite field $\mathbb{F}_{41}$ is 102,772. Since $41 \equiv 1 \bmod 8$, all the double points as well as the rulings on the fibers are rational. Hence blowing up of the 16 double points adds $16(p+1)^2 - 16 = 16(2p + p^2)$, $p = 41$ points. All the 32 special points on the conics in the singular locus are rational as are the 2 components in the fibers over the special points and the 3 components in the fibers over the 4 intersection points. It follows that blowing up the conics adds $4(3p+1) + 32(2p+1) + (4p - 32 - 4)(p+1) - 4p = 40 - 4p^2$, $p = 41$ points. Hence $\# Y'(\mathbb{F}_{41}) = 139,344$.

On the other hand since H^2 is spanned by divisors which are all rational over $\mathbb{F}_{41}$, frobenius acts by multiplication by p on H^2 and so we have

$$\begin{aligned} \# Y'(\mathbb{F}_{41}) &= t_0(41) + t_2(41) - t_3(41) + t_4(41) + t_6(41) \\ &= 1 + p(1+p)h^2 - t_3(41) + p^3, \qquad p = 41. \end{aligned}$$

Since $|t_3(41)| \leqslant h^3 p^{2/3}$ we get $|(1+p(1+p)h^2+p^3) - \# Y'(\mathbb{F}_{41})| \leqslant (2h^2-78)p^{2/3}$. This estimate gives $h^2=41$ so $h^3=4$ and $t_3(41)=180$.

Next we want to determine the L-function of the 4-dimensional Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(H^3(Y', \mathbb{Q}_l))$$

We need to know the trace of frobenius $t_3(p)$ for several primes.

PROPOSITION 2.9. *Let* $N_p = \# Y(\mathbb{F}_p)$ *then*

$$t_3(p) = (1 - 31p + 21p^2 + p^3) - N_p, \qquad p \equiv 1 \bmod 8$$

$$t_3(p) = (1 + 9p - 3p^2 + p^3) - N_p, \qquad p \equiv 3, 5 \bmod 8$$

$$t_3(p) = (1 - 7p - 5p^2 + p^3) - N_p, \qquad p \equiv 7 \bmod 8$$

Proof. In order to prove these formulas we need to know the rationality properties of the exceptional fibers in the blowing up $Y' \to Y$.

We first consider the isolated singularities:

The quadric surfaces over the points $(\pm\sqrt{2}, 0, 0, \pm\sqrt{2}, 1, 1, 0, 0)$ are given by the equations $U_0^2 - U_1^2 = \pm 4\sqrt{2}U_2U_3$. Over the points $(\pm\sqrt{-2}, 0, 0, \pm\sqrt{-2}, 0, 0, 1, 1)$ the equations are $U_0^2 + U_1^2 = \pm 4\sqrt{-2}U_2U_3$. The fibers over the points $(\pm\sqrt{2}, 0, 0, \pm\sqrt{-2}, 0, 0, 1, 1)$ have equations $U_0^2 - U_1^2 = \pm 4\sqrt{2}U_2U_3$ while the fibers over $(\pm\sqrt{-2}, 0, 0, \pm\sqrt{2}, 0, 0, 1, -1)$ have equations $U_0^2 + U_1^2 = \pm 4\sqrt{2}U_2U_3$.

Locally around each of the 4 intersection points

$$P_1 = (0:0:0:0:1:0:0:0),$$

$$P_2 = (0:0:0:0:0:1:0:0),$$

$$P_3 = (0:0:0:0:0:0:1:0),$$

$$P_4 = (0:0:0:0:0:0:0:1)$$

we have the following local equations

P_1:

$$UY_0^2 - VY_3^2 - WY_1^2Y_2^2 = 0$$

with

$$U = (16 - (Y_0^2 + Y_3^2)^2)^2 + 64Y_1^2 + 64Y_2^2$$

$$V = (16 - (Y_0^2 + Y_3^2)^2)^2 - 64Y_1^2 - 64Y_2^2$$

$$W = 16(16 - (Y_0^2 + Y_3^2)^2).$$

The fibers over the line $Y_0 = Y_2 = Y_3 = 0$ are given by the $(4 + Y_1^4)\, U_0^2 - (4 - Y_1^4)\, U_1^2 = 4 Y_1^2 U_2^2$ except over P_1 where we have the extra component $V_1^2 + V_2^2 = V_2^2$.

P_2:

Same local equations as above. The fibers over the line $Y_0 = Y_1 = Y_3 = 0$ are given by the equations $(4 + Y_2^4)\, U_0^2 - (4 - Y_2^4)\, U_1^2 = 4 Y_2^2 U_2^2$ except over P_2 where we have the extra component $V_0^2 + V_1^2 = V_2^2$.

P_3:

$$UY_0^2 + VY_3^2 - WY_1^2 Y_2^2 = 0$$

with

$$U = (16 - (Y_0^2 - Y_3^2)^2)^2 + 64 Y_1^4 + 64 Y_2^4$$
$$V = (16 - (Y_0^2 - Y_3^2)^2)^2 - 64 Y_1^4 - 64 Y_2^4$$
$$W = 16(16 + (Y_0^2 - Y_3^2)^2).$$

The fibers over the line $Y_0 = Y_2 = Y_3 = 0$ are given by $(4 + Y_1^4)\, U_0^2 + (4 - Y_1^4)\, U_1^2 = 4 Y_1^2 U_2^2$. Over P_3 we have the extra component $V_0^2 - V_1^2 = V_2^2$.

P_4:

Same equation as above. The fibers over the line $Y_0 = Y_1 = Y_3 = 0$ are given by $(4 + Y_2^4)\, U_0^2 + (4 - Y_2^4)\, U_1^2 = 4 Y_2^2 U_2^2$. Over P_4 we have the extra component $V_0^2 - V_1^2 = V_2^2$.

Let $p \equiv 1 \bmod 8$. Then everything is rational over $\mathbb{F}_p$ and the argument is exactly the same as in the proof of Proposition 2.8.

Next consider the case $p \equiv 3 \bmod 8$. There are 4 of the isolated double points rational over $\mathbb{F}_p$: $(\pm\sqrt{-2} : 0 : 0 : \pm\sqrt{-2} : 1 : -1 : 0 : 0)$ and the exceptional fibers have equations $U_0^2 + U_1^2 = \pm 4\sqrt{-2}\, U_2 U_3$. The discriminant is 8 which is not a square in $\mathbb{F}_p$ so the rulings are irrational and each fiber contributes $(p^2 + 1)$ points. Over the intersection points P_1 and P_2 all 3 components of the exceptional fibers are rational so the contribution from each of P_1 and P_2 is $3p + 1$. Over P_3 and P_4 the components of $U_0^2 + U_1^2 = 0$ are irrational so the only rational points in the fiber are those on the conic $V_0^2 - V_1^2 = 0$, thus P_3 and P_4 each contributes $p + 1$ rational points. Finally each of the 4 conics in the singular locus has 2 rational special points, the 2 solutions to $4 - Y^4 = 0$. The fibers are given by $8U_2 = 4 Y_2^2 U_2^2$. The two components are irrational over $\mathbb{F}_p$ so each special point only contributes 1 point. It follows that

$$\# Y'(\mathbb{F}_p) = N_p - \#\mathrm{Sing}\; Y(\mathbb{F}_p) + 4(p^2 + 1)$$
$$+ 2(3p + 1) + 2(p + 1) + 8 + (4p - 12)(p + 1).$$

The last term is the contribution from the smooth fibers over the 4 conics. Since the singular locus consists of $4 + 4p$ rational points we get

$$\# Y'(\mathbb{F}_p) = N_p - 4p + 8p^2.$$

Everything is rational over $\mathbb{F}_{p^2}$ so Frob_p^2 acts by multiplication by p^2 on $H^2(Y', \mathbb{Q}_l)$. Hence the eigenvalues of Frob_p are $\pm p$. Let a denote the multiplicity of $+p$ and b the multiplicity of $-p$ then a, b only depend on p mod 8. If $k = a - b$ then $t_2(p) = kp$ and $t_4(p) = kp^2$ thus we have the formula $t_3(p) = 1 + (k+4)\,p + (k-8)\,p^8 + p^3 - N_p$.

To determine k we compute $N_{11} = 1068$ so $t_3(11) = 132 - 660$. Since $|132k - 660| = |t_3(11)| \leqslant 4(11)^{3/2} \sim 145.9$ we get $3 < k < 6$. We have $a + b = 41$ so k must be odd hence $k = 5$ and the formula follows.

When $p \equiv 5$ mod 8 none of the isolated double points are rational. All components in the fibers over the intersection points are rational. On each of the 4 conics there are 4 rational special points but the components of the fibers are irrational. Counting as above we get

$$\# Y'(\mathbb{F}_p) = N_p - 8p - 4p^2.$$

It follows that $t_3(p) = 1 + (k+8)\,p + (k-4)\,p^2 + p^3 - N_p$. To determine k we compute $N_5 = 84$ and proceeding as above we get $k = 1$.

Finally when $p \equiv 7$ mod p there are 4 rational isolated double points: $(\pm\sqrt{2} : 0 : 0 : \sqrt{2} : 1 : 1 : 0 : 0)$. The quadrics in the fibers are given by the equations $U_0^2 - U_1^2 = \pm 4\sqrt{2}U_2 U_3$ hence the discriminant is 8 and the rulings are irrational. The fibers over P_1 and P_2 have 3 rational components and the fibers over P_3 and P_4 only have 1 rational component. Each of the conincs have 2 rational special points and the 2 components over a special point are both rational. It follows that

$$\# Y'(\mathbb{F}_p) = N_p + 20p + 8p^2$$

This gives the formula $t_3(p) = 1 + (k-20)\,p + (k-8)\,p + p^3 - N_p$. To determine k we compute $N_7 = 540$ and proceeding as above we get $k = 13$.

Let χ denote the Hecke character of $\mathbb{Q}[i]$ defined by $\chi(\mathfrak{p}) = a$ where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}[i]$ not dividing 2 and a is a generator of $\mathfrak{p}$ with $a \equiv 1 \bmod (2+2i)$. We identify χ with the orresponding character of $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q}[i])$. Let $\chi_{(-1)}$ denote the Tate twist, i.e., $\chi_{(-1)}(\mathfrak{p}) = Nm(\mathfrak{p})\,\chi(\mathfrak{p})$. Define a Galois representation

$$\sigma' : G' = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[i]) \to GL_4(\mathbb{Q}_2[i])$$

by $\sigma' = \mathrm{diag}(\chi^3, \chi_{(-1)}, \bar{\chi}^3, \bar{\chi}_{(-1)})$. We let ρ' denote the restriction of ρ:

$$\rho' : G' \to GSp(H^3(Y', \mathbb{Q}_2) \otimes \mathbb{Q}_2[i])$$

with the symplectic structure defined by cup-product.

THEOREM 2.10. (i) *The semi-simplifications of ρ' and σ' are isomorphic.*

(ii) *The semi-simplification of ρ is isomorphic to* $\mathrm{Ind}_{G'}^{G}(\chi^3 \oplus \chi_{(-1)})$.

(iii) $L(\rho, s) = L(\chi^3, s)\, L(\chi, s-1)$.

Proof. We use an automorphism of Y' to split the representation ρ' into a sum of 2-dimensional representations and then as before use the method from [Li] to identify these representations.

Let $[i]$ denote the automorphism of Y defined by

$$(Y_0 : Y_1 : Y_2 : Y_3 : X_0 : X_1 : X_2 : X_3)$$
$$\mapsto (Y_0 : Y_2 : Y_1 : iY_3 : X_2 : X_3 : X_0 : X_1).$$

This clearly lifts to an automorphism of the resolution Y' and acts by multiplication by i on $H^0(Y', \Omega^3)$. Thus $[i]^2$ is not the identity on $H^3(Y', \mathbb{Q}_2)$.

We consider the two cases:

If $[i]^2 \neq -I$ the subspaces $\ker([i]^2 + I)$ and $\ker([i]^2 - I)$ are 2-dimensional Galois stable subspaces of $H^3(Y', \mathbb{Q}_2)$ and hence ρ itself splits into a direct sum of 2-dimensional representations.

If $[i]^2 = -I$ (this is in fact the case that occurs though we shall not prove this here) we consider the subspaces $\ker([i] - i)$ and $\ker([i] + i)$ of $H^3(Y', \mathbb{Q}_2[i])$ hence we may assume that $\mathrm{Im}(\rho') \subset \{(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}) \mid A, D \in GL_2(\mathbb{Z}_2[i])\}$.

We shall give the proof of (i) only in the second case, the proof in the first case is similar and easier.

Let r denote the representation of G' on $\ker([i] - i)$. The eigenvalues of the frobenius elements $F(\mathfrak{p})$ at the primes $\mathfrak{p}$ in $\mathbb{Z}[i]$ lying over 3 and 5 can be determined from the number of points on Y' over the fields $\mathbb{F}_3, \mathbb{F}_9, \mathbb{F}_5$, and $\mathbb{F}_{25}$. We get -27 with multiplicity 4 for the prime 3 and for each of the primes lying over 5 we get $(1 \pm 2i)^3$ and $5(-1 \pm 2i)$. It is clear that $\ker([i] - i)$ and $\ker([i] + i)$ are isotropic subspaces and hence we can choose a basis of eigenvectors e_1, e_2 for $\ker([i] - i)$ and e_3, e_4 for $\ker([i] + i)$ such that $\{e_1, e_2, e_3, e_4\}$ is a symplectic basis for $H^3(Y', \mathbb{Q}_2[i])$. Assume that $r(F(-1+2i))e_1 = (-1+2i)^3 e_1$. Then $\rho'(F(-1+2i))\, e_3 = (-1-2i)^3\, e_3$ and so $r(F(-1+2i))\, e_2 = 5(-1 \pm 2i))\, e_2$. The determinant of r is a Hecke character δ of $\mathbb{Q}[i]$, unramified except at the prime over 2, taking values in $\mathbb{Z}[i]$. It follows that δ factors through $\mathbb{C}^\times \times (\mathbb{Z}[i]_{(1+i)}^\times / \{\pm 1, \pm i\} \simeq \mathbb{C}^\times \times U_3/U_7$ where $U_n = \{x \in \mathbb{Z}[i]_{(1+i)}^\times \mid x \equiv 1 \bmod (2+2i)^n\}$. The logarithm defines a isomorphism $U_3 \simeq \mathbb{Z}[i]_{(1+i)}$ and $U_3/U_7 \simeq (\mathbb{Z}/4)^2$. Thus δ is a twist of either $\chi^3 \chi_{(-1)}$ or $\chi^3 \bar{\chi}_{(-1)}$. By computing traces for some other primes we find that the twist is trivial.

By class field theory $\mathbb{Q}[i]$ has 3 quadratic extensions, ramified only at the prime $(1+i)$, namely $\mathbb{Q}[\sqrt{i}]$, $\mathbb{Q}[\sqrt{1+i}]$ and $\mathbb{Q}[\sqrt{1-i}]$. None of these have a cubic extension unramified outside the prime over 2. Now by reducing modulo $(1+i)r$ defines a representation in $GL(2, \mathbb{F}_2) \simeq S_3$. The considerations above shows as before that $\operatorname{Im} r \subset \mathbb{Z}/2$ and hence $\operatorname{Tr} r(g) \equiv 0 \bmod 1+i$ for all $g \in G'$. Again applying Theorem 4.2 of [Li] we get $r \simeq \chi^3 \oplus \chi_{(-1)}$ or $\chi^3 \oplus \bar{\chi}_{(-1)}$. This proves (i), (ii), and (iii) are straightforward consequences.

Remark. The L-function of $H^3(Y')$ also occurs as a factor in the L-function of $H^3(E \times E \times E)$ where $E: y^2 = 1 + x^4$ is the elliptic curve with CM by $\mathbb{Q}[i]$. The Tate conjecture then predicts the existence of a correspondence between Y' and $E \times E \times E$. We shall exhibit such a correspondence.

Consider the genus 5 curve C given by the equations

$$u^2 = 1 - t^4$$
$$v^2 = 1 + t^4$$

and two copies of the elliptic curve E

$$p^2 = 1 + a^4$$
$$q^2 = 1 + b^4.$$

We define a rational map $\phi : C \times E \times E \to Y$ by

$$\begin{aligned} Y_0 &= \sqrt{2}vab & X_0 &= 1 \\ Y_1 &= \sqrt{2}tbp & X_1 &= a^2b^2 \\ Y_2 &= \sqrt{2}taq & X_2 &= a^2t^2 \\ Y_3 &= \sqrt{2}uab & X_3 &= b^2t^2. \end{aligned}$$

Mapping $C \times E \times E \to E \times E \times E$ by $\psi : ((u, v, t), (p, a), (q, b)) \mapsto ((uv, t^2), (p, a), (q, b))$ we obtain a correspondence.

Consider the rational 3-form ω on Y defined by $dX_1\, dX_2\, dX_3/Y_0 Y_1 Y_2 Y_3$. It is easy to see that this extends to a holomorphic 3-form on the resolution Y'. Pulling back by ϕ we get $\phi^*\omega = -16((abt)^3/4uva^3b^3t^2pq)\, da\, db\, dt = -4(t/uvpq)\, da\, db\, dt$. The holomorphic 3-form on $E \times E \times E$ is given by $dx\, da\, db/ypq$ where x, y are the coordinates on the first copy of E. Pulling back this 3-form by ψ we get $-4\psi^*(dx\, da\, db/ypq) = \phi^*\omega$. It follows that the correspondence induces an isomorphism $H^{3,0}(Y') \simeq H^{3,0}(E \times E \times E)$.

Finally we consider the orbit of the 6-tuple

$$(0000, 0001, 0010, 0011, 0110, 0100).$$

From the theta relations we get the complete intersection Z defined by the equations

$$Y_0^2 = 2(X_0X_3 + X_1X_2)$$

$$Y_1^2 = 2(X_0X_3 - X_1X_2)$$

$$Y_2^2 = 2(X_0X_2 - X_1X_3)$$

$$Y_3^2 = 2(X_0X_2 + X_1X_3).$$

The Euler characteristic can be computed by viewing Z as a sequence of 4 $2:1$ coverings.

LEMMA 2.11. $\chi(Z) = 4$.

Proof. Similar to the proof of Lemma 2.6.

Next we determine the singular locus of Z and construct a resolution $Z' \to Z$.

PROPOSITION 2.12. (i) *The singular locus consists of the two non-intersecting lines*

$$L_1 : Y_0 = Y_1 = Y_2 = Y_3 = X_0 = X_1 = 0$$

$$L_2 : Y_0 = Y_1 = Y_2 = Y_3 = X_2 = X_3 = 0$$

(ii) *The blowing up of these lines, Z' is a smooth 3-fold.*

(iii) *Let $\pi : Z' \to Z$ denote the blow-up. Then the restriction $\pi : \pi^{-1}(L_i) \to L_i$ $i = 1, 2$ defines an elliptic fibration which is the universal family of elliptic curves with level 4 structure.*

(iv) $\chi(Z') = 48$.

Proof. (i) This follows easily, noting that a singular point must have at least 2 of its Y-coordinates 0 and that the 3 automorphisms defined by

$X_1 \mapsto -X_1$ the other X_j fixed

$X_1 \mapsto iX_1$, $X_2 \mapsto iX_2$ the other X_j fixed

$X_0 \mapsto X_0 - X_1$, $X_1 \mapsto X_0 + X_1$, $X_2 \mapsto X_2 - X_3$, $X_3 \mapsto X_2 + X_3$

generate all permutations of the Y-coordinates.

(ii) Consider the open piece where $X_3=1$. In this open affine the equations become

$$4X_0 = Y_0^2 + Y_1^2$$
$$4X_1 = -Y_2^2 + Y_3^2$$
$$4X_0X_2 = Y_2^2 + Y_3^2$$
$$4X_1X_2 = Y_0^2 - Y_1^2.$$

Substituting the first two equations in the last two, the 3-fold, as a subvariety of $\mathbb{A}^5$, is defined by the equations

$$(Y_0^2 + Y_1^2)X_2 = Y_2^2 + Y_3^2$$
$$(-Y_2^2 + Y_3^2)X_2 = Y_0^2 - Y_1^2.$$

Blowing up the line $Y_0 = Y_1 = Y_2 = Y_3 = 0$ we get the equations

$$(Z_0^2 + Z_1^2)X_2 = Z_2^2 + 1$$
$$(-Z_2^2 + 1)X_2 = Z_0^2 - Z_1^2.$$

where $Y_0 = Z_0 Y_3$, etc.

A global model is given by

$$(Z_0^2 + Z_1^2)X_2 = (Z_2^2 + Z_3^2)X_1$$
$$(-Z_2^2 + Z_3^2)X_2 = (Z_0^2 - Z_1^2)X_1.$$

in $\mathbb{P}^3 \times \mathbb{P}^1$ with coordinates (Z_0, Z_1, Z_2, Z_3), (X_1, X_2). It is immediate that the blow-up is smooth.

(iii) The fiber over a point on the line parametrized by X_2 is the intersection of 2 quadrics in $\mathbb{A}^3$ so for a general X_2 the fiber is an elliptic curve. To see explicitly that this family is the universal level 4 curve, recall that an elliptic curve defined by 2 quadrics, given by binary quadratic forms A and B, is the $2:1$ cover of $\mathbb{P}^1$ branching over the points $(s:t)$ where $\det(sA+tB)=0$. In our case the branch locus is given by $(X_2^2 - s^2)(X_2^2 s^2 - 1) = 0$ or equivalently, $(s^2 - X_2^2)(s^2 - X_2^{-2}) = 0$. This model is the same as the one given in [Sh] for the universal level 4 curve.

(iv) The elliptic modular surface of level 4 is a K3 surface so the Euler characteristic is 24 and hence the Euler characteristic of Z' is $4 - 4 + 2 \times 24 = 48$.

Z is in fact birational to $\mathbb{P}^1 \times F$ where F is a Fermat quartic surface in $\mathbb{P}^3$ defined by $Z_0^4 - Z_1^4 + Z_2^4 - Z_3^4 = 0$. The existence of a rational map $Z \to F$ follows from a classical theta relation.

Proposition 2.13. *The rational map $\phi : \mathbb{P}^1 \times F \to Z \subset \mathbb{P}^7$, defined on a suitable open subset by*

$$\begin{aligned}\phi(t, Z_0, Z_1, Z_2, Z_3) &\\ &= (cZ_0 : cZ_1 : cZ_2 : cZ_3 : \\ &\quad (Z_0^2 + Z_1^2)/t : (-Z_2^2 + Z_3^2)/t : t(Z_2^2 + Z_3^2)/(Z_0^2 + Z_1^2) : t)\end{aligned}$$

with $c^2 = 2$, is birational.

Proof. One verifies directly that ϕ maps into Z and it is clear that it has degree 1 and is dominant.

Proposition 2.14. *The Betti and Hodge numbers of Z' are given by*

$$h^0 = h^6 = 1$$
$$h^1 = h^5 = 0$$
$$h^2 = h^4 = 23$$
$$h^3 = 0$$
$$h^{2,0} = 1, \qquad h^{1,1} = 21.$$

Proof. Since Z' is a Siegel modular 3-fold it has $h^1 = 0$ and so

$$h^2 - h^3 + h^4 = 2h^2 - h^3 = 46.$$

Since Z' is birational to $\mathbb{P}^1 \times F$ we have $h^{3,0} = 0$ and $h^{2,0} = 1$. The Hodge structure on $H^2(F)$ splits over $\mathbb{Q}$ and all of $H^{1,1}$ is algebraic. It follows that the Hodge structure on $H^2(Z')$ splits over $\mathbb{Q}$ and so $H^{1,1}(Z')$ is spanned by algebraic classes. Hence the Galois representation on $H^2_l(Z')$ is a direct sum of a 2-dimensional "transcendental" part and a $h^{1,1}(Z')$-dimensional "algebraic" part. On the algebraic part the Frobenius element at p has eigenvalues $p \times$ (a root of unity).

If $p \equiv 1 \bmod 8$ the algebraic part is rational over $\mathbb{F}_p$ and so Frobenius acts by multiplication by p. We compute $\# Z(\mathbb{F}_{17}) = 9636$. The number of $\mathbb{F}_{17}$-rational points on the universal level 4 family is 600. Hence we get $\# Z'(\mathbb{F}_{17}) = 9636 - 2 \times 18 + 2 \times 600 = 10800$.

On the other hand

$$\begin{aligned}\# Z'(\mathbb{F}_{17}) &= t_0(p) + t_2(p) - t_3(p) + t_4(p) + t_6(p) \\ &= 1 + p(1 + p)\, h^{1,1} + t(1 + p) - t_3(p) + p^3\end{aligned}$$

where t is the trace of Frobenius on the 2-dimensional trancendental part of H^2.

The representation on the transcendental part of $H^2(Z')$ is isomorphic to the representation on the transcendental part of $H^2(F)$. The Galois representation on the transcendental part of the Fermat quartic is given by $\chi^2 \oplus \bar{\chi}^2$ where χ is the same Hecke character as in the previous section. It follows that $t = (1+4i)^2 + (1-4i)^2 = -30$. Since $|t_3(p)| \leqslant h^3 p^{2/3}$ we get the estimate

$$|1 + p(1+p)\,h^{1,1} + t(1+p) + p^3 - \#Z'(\mathbb{F}_{17})|$$
$$= |t_3(p)| \leqslant h^3 p^{2/3} = (2h^{1,1} - 42)\,p^{2/3}$$

It follows from this estimate that $h^{1,1} = 21$, $h^2 = 23$ and $h^3 = 0$.

APPENDIX: Elliptic Modular Forms Associated to Siegel Modular Forms

In this appendix we apply some results of Shintani [Sh] and Oda [Od] to sketch a proof that the Siegel modular form considered in the first part of section 2 is a Saito–Kurokawa lift and to compute its Andrianov L-function.

The result is the following:

THEOREM. *Let ϕ be one of the fifteen products of six theta constants ssociated to a 6-tuple of distinct theta characteristics with sum $= 0$ (these form a basis of eigenforms for the space of cusp forms of weight 3 for the principal congruence subgroup $\Gamma(4) \subset Sp_4(\mathbb{Z})$). Then the Andrianov L-function is of the form*

$$L(\phi, s) = L(\chi, s-1)\, L(g, \chi, s)\, L(\chi, s-2),$$

where g is the unique normalized eigenform in $S_4(\Gamma_0(8))$, χ is a Dirichlet character mod 4, and $L(\chi, s)$ denotes the Dirichlet L-function.

We consider the space V of 4×4 skew-symmetric matrices

$$\begin{pmatrix} \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & B \\ -{}^tB & \begin{matrix} 0 & c \\ -c & 0 \end{matrix} \end{pmatrix}$$

with $\mathrm{Tr}(B) = 0$. The symplectic group $Sp_4(\mathbb{R})$ acts on V as follows: $\sigma \in Sp_4(\mathbb{R})$ sends $M \in V$ to $\sigma M {}^t\sigma$. We define a quadratic form Q^* on V by $Q^*(M) = 2ac - 2\det(B)$. The action of $Sp_4(\mathbb{R})$ preserves this form and in fact we get an isomorphism $PSp_4(\mathbb{R}) = Sp_4(\mathbb{R})/\pm I \simeq SO^0(Q^*)$. Let e_{ij},

$i<j$ denote the skew-symmetric matrix with 1 as the i, j'th entry, -1 as the j, i'th entry and 0's everywhere else. Then V is spanned by $\{e_{12}, e_{14}, e_{13}-e_{24}, e_{23}, e_{34}\}$. The matrix of the quadratic form Q^* with respect to this basis is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so it has signature $(+3, -2)$. Let $Q=-2Q^{*-1}$ so Q has matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and signature $(+2, -3)$. Consider a positive rational multiple Q' of Q^* and let $\mathscr{L}$ be a lattice in V such that Q' takes integral values on $\mathscr{L}$. Let $\mathscr{L}^*$ be the dual lattice with respect to Q' and let Γ' be a congruence subgroup of $Sp_4(\mathbb{Z})$ which leaves $\mathscr{L}$ stable. Then Γ' acts on the finite group $\mathscr{L}^*/\mathscr{L}$. Let ξ be a character of Γ' such that $\ker \xi$ is also a congruence subgroup. Consider a form $\phi \in S_k(\Gamma', \xi)$, i.e., a cusp-form satisfying $\phi(\sigma\tau)=\xi(\sigma)\det(\gamma\tau+\delta)^k\,\phi(\tau)$ where $\sigma=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\Gamma'$. Consider a vector $h\in\mathscr{L}^*$ with $Q'(h)>0$. The stabilizer G_h of h in $PSp_4(\mathbb{R})$ is isomorphic to the Lorenz group $SO^0(1,3)$ and the homogeneous space $X_h=G_h/K_h$, where $K_h=G_h\cap K$, K a maximal compact subgroup of $PSp_4(\mathbb{R})$, is hyperbolic 3-space embedded in the Siegel space $\mathfrak{S}_2=PSp_4(\mathbb{R})/K$. The volume form $\omega=dt\wedge dz\wedge dw$ for $\tau=\left(\begin{smallmatrix}t & z\\ z & w\end{smallmatrix}\right)\in\mathfrak{S}_2$ restricts to a volume form on X_h which we shall also denote by ω. Let $\phi\in S_3(\Gamma',\xi)$ then $\phi\omega$ transforms under Γ' by $\sigma^*(\phi\omega)=\xi(\sigma)\,\phi\omega$. Let ν be a function on the finite set $\mathscr{L}^*/\mathscr{L}$ such that $\nu(\sigma x)=\xi^{-1}(\sigma)\,\nu(x)$. Remark that the existence of such a function forces ξ to be trivial on the isotropy subgroup $\Gamma_h\subset\Gamma'$ for any h with $\nu(h)\neq 0$ and hence the integral $I(\phi,h)=\int_{X_h/\Gamma'_h}\phi\omega$ is well-defined. We call this integral the period integral of ϕ with respect to h. Remark also that $\nu(h)\,I(\phi,h)$ only depends on the Γ'-orbit of h.

For $z\in\mathfrak{h}$, $\mathfrak{h}$ the usual upper half-plane, consider the sum

$$\Theta(\phi,\nu)=\sum_{\{h\}}\nu(h)\,\mathbf{e}(\tfrac{1}{2}Q'(h)z))\,I(\phi,h),$$

where we sum over the Γ'-orbits $\{h\}$ in $\mathscr{L}^*$ with $Q'(h) > 0$ and where $\mathbf{e}(-)$ denotes $\exp(2\pi i -)$.

For each $k \in \mathscr{L}^*/\mathscr{L}$ we let $\Theta(\phi, k)$ denote the sum

$$\sum_{\{h\} \text{ with } h \equiv k \bmod \mathscr{L}} \mathbf{e}(\tfrac{1}{2}Q'(h)z)\, I(\phi, h)$$

so $\Theta(\phi, v) = \sum_{\{k\} \in \mathscr{L}^*/\mathscr{L}} v(k)\, \Theta(\phi, k)$. The following is a special case of a result due to Shintani:

THEOREM A.1. *Let $\phi \in S_3(\Gamma', \xi)$. Then $\Theta(\phi, v)$ is a holomorphic function on $\mathfrak{h}$ which vanishes at $i\infty \cup \mathbb{Q}$.*

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $abQ'(x) \equiv cdQ'(y) \equiv 0 \bmod 2$ for all $x, y \in \mathscr{L}$ and such that $cQ'(h) \equiv 0 \bmod 2$ for all $h \in \mathscr{L}^*$. Then

$$\Theta(\phi, v)(\gamma z) = \begin{cases} \displaystyle\sum_{k \in \mathscr{L}^*/\mathscr{L}} v(dk)\, \mathbf{e}\left(\frac{ab}{2} Q'(k)\right) \Theta(\phi, k)(z) & \text{for } c = 0 \\ \displaystyle\varepsilon_d^5 \left(\frac{-2c}{d}\right)\left(\frac{D}{d}\right)(cz+d)^{5/2} & \\ \displaystyle\quad \times \sum_{k \in \mathscr{L}^*/\mathscr{L}} v(dk)\, \mathbf{e}\left(\frac{ab}{2} Q'(k)\right) \Theta(\phi, k)(z) & \text{for } d > 0 \\ \displaystyle\varepsilon_d^{-5} \left(\frac{2c}{d}\right)\left(\frac{D}{-d}\right)(cz+d)^{5/2} & \\ \displaystyle\quad \times \sum_{k \in \mathscr{L}^*/\mathscr{L}} v(dk)\, \mathbf{e}\left(\frac{ab}{2} Q'(k)\right) \Theta(\phi, k)(z) & \text{for } d < 0, \end{cases}$$

where D is the discriminant of $\mathscr{L}$ and $\varepsilon_d = 1$ if $d \equiv 1 \bmod 4$ and $\varepsilon_d = i$ if $d \equiv 3 \bmod 4$.

For the proof we refer to [Sh, Proposition 1.6 and Proposition 1.7].

In our application of Shintani's theorem we take Γ' to be the principal congruence subgroup $\Gamma(2)$. This group is generated by the following 10 matrices:

$$A_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$B_{11}=\begin{pmatrix}1&0&2&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad B_{12}=\begin{pmatrix}1&0&0&2\\0&1&2&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad B_{22}=\begin{pmatrix}1&0&0&0\\0&1&0&2\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$C_{11}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&0&1&0\\0&0&0&1\end{pmatrix}\quad C_{12}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{pmatrix}\quad C_{22}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&2&0&0\end{pmatrix}.$$

Let ξ be the character on $\Gamma(2)$ defined on the generators by $\xi(A_{11})=\xi(A_{12})=\xi(A_{21})=\xi(A_{22})=\xi(B_{11})=\xi(B_{22})=\xi(C_{11})=\xi(C_{22})=-1$ and $\xi(B_{12})=\xi(C_{12})=1$. It was shown in [Ny] that $S_3(\Gamma(2),\xi)$ is 1-dimensional, spanned by the product

$$\phi=\theta_{0001}\theta_{0010}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}$$

Since the sum of the theta characteristics is 0, ϕ is one of the 15 forms in the first of the orbits considered in section 2.

Let L be the lattice of integral matrices in V and put $Q'=\frac{1}{8}Q$ so Q' has matrix

$$\begin{pmatrix}0&0&0&0&-\frac{1}{4}\\0&0&0&-\frac{1}{4}&0\\0&0&-\frac{1}{8}&0&0\\0&-\frac{1}{4}&0&0&0\\-\frac{1}{4}&0&0&0&0\end{pmatrix}.$$

It follows that Q' takes integral values on the lattice $\mathscr{L}$ spanned by $\{4e_{12},4e_{14},8(e_{13}-e_{24}),4e_{23},4e_{34}\}$. The dual lattice $\mathscr{L}^*$ is equal to L and $\mathscr{L}^*/\mathscr{L}\simeq\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/8\times\mathbb{Z}/4\times\mathbb{Z}/4$. Clearly $\Gamma(2)$ stabilizes $\mathscr{L}$ so we get an action of $\Gamma(2)$ on $L/\mathscr{L}$.

Consider the element $v_1\in L$ represented by the matrix

$$\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$$

so v_1 has coordinates $(0,1,0,-1,0)$ with respect to the basis $\{e_{12},e_{14},(e_{13}-e_{24}),e_{23},e_{34}\}$. We define a function ν on $L/\mathscr{L}$ by $\nu(gv_1)=\xi(g)$ on the orbit of v_1 and ν is identically 0 on the complement of the orbit. To verify that ν is well-defined it suffices to show that ξ is trivial on the isotropy subgroup $\Gamma(2)_{v_1} \subset \Gamma(2)$.

Since ξ is trivial on the congruence subgroup $\Gamma(4)$ this equivalent to showing that ξ is trivial on $\Gamma(2)_{v_1}/(\Gamma_{v_1} \cap \Gamma(4)) \subset \Gamma(2)/\Gamma(4) = (\mathbb{Z}/2)^{10}$. Also since $gv_1 \equiv v_1 \bmod \mathscr{L}$ implies $gv_1 \equiv v_1 \bmod 4L$ we have $\Gamma(2)_{v_1}/(\Gamma(2)_{v_1} \cap \Gamma(4)) \subset (\Gamma(2)/\Gamma(4))_{\bar{v}_1} =$ isotropy subgroup of $\Gamma(2)/\Gamma(4)$ of $\bar{v}_1 \in L/4L$. The next lemma shows that ξ is trivial on this group.

Lemma A.2. *The isotropy group* $(\Gamma(2)/\Gamma(4))_{\bar{v}_1} \subset (\mathbb{Z}/2)^{10}$ *is generated by the* 6 *elements* $A_{11}A_{22}$, $A_{12}A_{21}$, B_{12}, C_{12}, $B_{11}B_{22}$, $C_{11}C_{22}$.

Proof. The generators of $\Gamma(2)$ act on v_1 as follows

$$\begin{array}{ll} A_{11}: v_1 \mapsto (0, -1, 0, 1, 0) & A_{12}: v_1 \mapsto (0, 5, -2, -1, 0) \\ A_{21}: v_1 \mapsto (0, 1, -2, -5, 0) & A_{22}: v_1 \mapsto (0, -1, 0, 1, 0) \\ B_{11}: v_1 \mapsto (2, 1, 0, -1, 0) & C_{11}: v_1 \mapsto (0, 1, 0, -1, 2) \\ B_{12}: v_1 \mapsto (0, 1, 0, -1, 0) & C_{12}: v_1 \mapsto (0, 1, 0, -1, 0) \\ B_{22}: v_1 \mapsto (2, 1, 0, -1, 0) & C_{22}: v_1 \mapsto (0, 1, 0, -1, 2). \end{array}$$

Let H be the subgroup generated by the 6 elements above. The quotient $(\Gamma(2)/\Gamma(4))/H$ is generated by A_{11}, A_{12}, B_{11}, C_{11} and using the description above one can check directly that no element in this group satisfies $gv_1 \equiv v_1 \bmod 4$.

Lemma A.3. *The isotropy group* $G_{v_1}(\mathbb{R})$ *is the subgroup of* $Sp_4(\mathbb{R})$ *consisting of matrices of the form*

$$\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & b_2 & -a_2 \\ a_3 & b_3 & a_4 & b_4 \\ b_3 & -a_3 & -b_4 & a_4 \end{pmatrix}.$$

Proof. Let J denote the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ so $v_1 = \left(\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right)$. A symplectic matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_{v_1}$ if and only if the following equations are satisfied: $AJ'B + BJ'A = 0$, $BJ'C + AJ'D = J$, $DJ'A + CJ'B = J$, $DJ'C + AJ'D = 0$. Put $A^* = -JAJ$, $B^* = JBJ$, $C^* = JCJ$, $D^* = -JDJ$. Then the above equations show that $\left(\begin{smallmatrix} A^* & B^* \\ C^* & D^* \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Since we also have $M\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right){}^tM = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ we get $M = \left(\begin{smallmatrix} A^* & B^* \\ C^* & D^* \end{smallmatrix}\right)$.

Corollary A.4. $G_{v_1}(\mathbb{R})$ *is isomorphic to* $SL_2(\mathbb{C})$.

Proof. The isomorphism is given by

$$\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & b_2 & -a_2 \\ a_3 & b_3 & a_4 & b_4 \\ b_3 & -a_3 & -b_4 & a_4 \end{pmatrix} \mapsto \begin{pmatrix} a_1+ib_1 & b_2+ia_2 \\ b_3-ia_3 & a_4-ib_4 \end{pmatrix}.$$

We now come to the main result of the appendix.

THEOREM A.5. *$\Theta(\phi, v)$ is a non-zero cusp form of weight $\frac{5}{2}$ for the group $\Gamma_0(16)=\{(\begin{smallmatrix} a & d \\ b & d \end{smallmatrix}) \in SL_2(\mathbb{Z}) \mid c \equiv 0(16)\}$ with character χ given by $\chi((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}))=$ $\mathbf{e}(ab/4)$, i.e., $\Theta(\phi, v)$ satisfies the transformation law*

$$\Theta(\phi, v)(\gamma z)=\chi(\gamma)\, \varepsilon_d^{-5} \left(\frac{c}{d}\right)^5 (cz+d)^{5/2}\, \Theta(\phi, v)(z),$$

where $\gamma=(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(16)$.

Proof. Observing that $dv_1 \equiv A_{11} v_1 \bmod 4$ if $d \equiv -1 \bmod 4$, and $\equiv v_1 \bmod 4$ if $d \equiv 1 \bmod 4$ we get $v(dk)=(-1/d)\, v(k)$ for all $k \in L/\mathscr{L}$. Since the discriminant of Q' on $\mathscr{L}$ is -8×16^2 and since $\varepsilon_d^{-1}=(-1/d)\,\varepsilon_d$ it follows immediately from Theorem A.1 that $\Theta(\phi, v)$ satisfies the transformation law above and that it is a cusp form i.e., $\Theta(\phi, v) \in S_{5/2}(\Gamma_0(16), \chi)$. We shall prove that it is non-zero.

The Fourier expansion of $\Theta(\phi, v)$ is given by

$$\Theta(\phi, v)(z)=\sum_h v(h)\, \mathbf{e}(\tfrac{1}{2} Q'(h) z)\, I(\phi, h).$$

Since $v(h)=0$ unless $h=gv_1+w$ for some $g \in \Gamma(2)$ and $w \in \mathscr{L}$, we only have terms for which $Q'(h)=Q'(gv_1)+2(gv_1, w)+Q'(w) \in \frac{1}{2}+2\mathbb{Z}$. It follows that we can write the Fourier expansion as

$$\Theta(\phi, v)(z)=\sum_{n \equiv 1(4)} c_n\, \mathbf{e}\left(\frac{n}{4} z\right),$$

where $c_n=\sum_{\{h\} \text{ with } 2Q'(h)=n} v(h)\, I(\phi, h)$.

It suffices to show that one of the c_n's is non-zero. We first show that if $v(h) \neq 0$ and $2Q'(h)=1$ then h is in the orbit of v_1. If $2Q'(h)=1$ then h must be primitive. Indeed if $h=nh'$ we would have $4=n^2 Q(h')$ so if $n \neq 1$, n must be 2 and $Q(h')=1$. But if $h'=(a, b, c, d, e)$ we have $1=Q(h')=-4ae-4bd-c^2$ so $c^2 \equiv -1 \bmod 4$, which is impossible. Now both v_1 and h are primitive elements in L with $Q(v_1)=Q(h)$. By a theorem of Humbert, this implies that v_1 and h are conjugate under $Sp_4(\mathbb{Z})$. Assume $h=g'v_1$ for some $g' \in Sp_4(\mathbb{Z})$. Since $v(h) \neq 0$ we also have

$g'v_1 = h = gv_1 + w$ for some $g \in \Gamma(2)$ and $w \in \mathscr{L}$. In particular we have $g'v_1 \equiv gv_1 \bmod 2$. The same argument as in the proof of Lemma A.3 shows that the isotropy subgroup of $\bar{v}_1 \in L/2L$ in $Sp_4(\mathbb{Z}/2)$ is isomorphic to $SL_2(\mathbb{Z}[i]/2)$. The reduction map $SL_2(\mathbb{Z}[i]) \to SL_2(\mathbb{Z}[i]/2)$ is surjective by strong approximation (or by a direct verification) and since $g^{-1}g'$ maps to an element in $SL_2(\mathbb{Z}[i]/2) \subset Sp_4(\mathbb{Z}/2)$ we can write $g^{-1}g' = \sigma\lambda$ with $\sigma \in \Gamma(2)$ and $\lambda \in \Gamma_{v_1}$. It follows that $h = g'v_1 = g\sigma\lambda v_1 = g\sigma v_1$ and $g\sigma \in \Gamma(2)$. This argument shows that $\{h \in L \mid 2Q'(h) = 1\} = \Gamma(2)v_1$ hence we get $c_1 = I(\phi, v_1)$ and thus it suffices to show that this period integral does not vanish.

Let K be the stabilizer in $Sp_4(\mathbb{R})$ of $iI_4 \in \mathfrak{S}_2$ so $K = U(2)$. Then $X_{v_1} = G_{v_1}/K_{v_1} = \{(\begin{smallmatrix} t & b \\ b & -\bar{t} \end{smallmatrix}) \in \mathfrak{S}_2 \mid t \in \mathfrak{h},\ b \in \mathbb{R}\}$. This 3-manifold can be identified with hyperbolic 3-space $\mathbb{C} \times \mathbb{R}_{>0}$ via the map $(\begin{smallmatrix} t & b \\ b & -\bar{t} \end{smallmatrix}) \mapsto (a + ib, r)$ where $t = a + ir \in \mathfrak{h}$.

Let $\iota : \mathfrak{S}_2 \to \mathfrak{S}_2$ denote the involution $(\begin{smallmatrix} t & z \\ z & w \end{smallmatrix}) \mapsto (\begin{smallmatrix} -\bar{w} & \bar{z} \\ \bar{z} & -\bar{t} \end{smallmatrix})$. Clearly X_{v_1} is the fixed point set of ι.

Consider a theta characteristic $\varepsilon\varepsilon'\delta\delta'$. We have

$$\begin{aligned}
\theta_{\varepsilon\varepsilon'\delta\delta'}(\iota(\tau)) &= \sum_{(p_1,\, p_2) \in \mathbb{Z}} \mathbf{e}\left(\frac{1}{2}\left(-\bar{w}\left(p_1 + \frac{\varepsilon}{2}\right)^2 + 2\bar{z}\left(p_1 + \frac{\varepsilon}{2}\right)\left(p_2 + \frac{\varepsilon'}{2}\right)\right.\right. \\
&\qquad \left.\left. - \bar{t}\left(p_2 + \frac{\varepsilon'}{2}\right)^2 + \left(p_1 + \frac{\varepsilon}{2}\right)\delta + \left(p_2 + \frac{\varepsilon'}{2}\right)\delta'\right)\right) \\
&= \overline{\sum \mathbf{e}\left(\frac{1}{2}\left(w\left(p_1 + \frac{\varepsilon}{2}\right)^2 - 2z\left(p_1 + \frac{\varepsilon}{2}\right)\left(p_2 + \frac{\varepsilon'}{2}\right)\right.\right.} \\
&\qquad \overline{\left.\left. + t\left(p_2 + \frac{\varepsilon'}{2}\right)^2 - \left(p_1 + \frac{\varepsilon}{2}\right)\delta - \left(p_2 + \frac{\varepsilon'}{2}\right)\delta'\right)\right)} \\
&= \overline{\sum \mathbf{e}\left(\frac{1}{2}\left(w\left(-p_1 - \frac{\varepsilon}{2}\right)^2 + 2z\left(-p_1 - \frac{\varepsilon}{2}\right)\left(p_2 + \frac{\varepsilon'}{2}\right)\right.\right.} \\
&\qquad \overline{\left.\left. + t\left(p_2 + \frac{\varepsilon'}{2}\right)^2 + \left(-p_1 - \frac{\varepsilon}{2}\right)\delta + \left(p_2 + \frac{\varepsilon'}{2}\right)(-\delta')\right)\right)} \\
&= \overline{\theta_{\varepsilon'\varepsilon\delta'-\delta}(\tau)} \\
&= \overline{\theta_{\varepsilon'\varepsilon\delta'\delta}(\tau)}.
\end{aligned}$$

Consider again the cusp form ϕ. If τ is a fixed point for ι we have

$$\begin{aligned}
\phi(\tau) = \phi(\iota(\tau)) &= \theta_{0001}(\tau)\, \theta_{0010}(\iota(\tau))\, \theta_{0100}(\tau)\, \theta_{1000}(\iota(\tau))\, \theta_{0110}(\tau)\, \theta_{1001}(\iota(\tau)) \\
&= |\theta_{0001}(\tau)\, \theta_{0100}(\tau)\, \theta_{0110}(\tau)|^2
\end{aligned}$$

Let $\mathscr{D}$ be a fundamental domain for the action of $\Gamma(2)_{v_1}$ on $X_{v_1} = \mathbb{C} \times \mathbb{R}_{>0}$ then $I(\phi, v_1) = \int_{\mathscr{D}} |\theta_{0001}(\tau)\theta_{0100}(\tau)\theta_{0110}(\tau)|^2 d\tau$. But $|\theta_{0001}(\tau)\theta_{0100}(\tau)\theta_{0110}(\tau)|^2$ is non-negative, continuous and not identically 0 (evaluate for instance at iI_4) on $\mathscr{D}$. Hence this integral is strictly positive so $c_1 \neq 0$.

Using the next two lemmas we easily obtain a formula for the other Fourier coefficients.

LEMMA A.6. *Let $h, k \in L$ with $Q'(h) = Q'(k) = d > 0$. Then h and k are conjugate under $\Gamma(2)$ if and only if they are conjugate under $Sp_2(\mathbb{Z})$ and* $h \equiv k \bmod 2$.

Proof. Assume $\sigma(h) = k$, $\sigma \in Sp_4(\mathbb{Z})$ and $h \equiv k \bmod 2$. Let $\mathcal{O}$ be the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. By [An] the isotropy subgroup $\Gamma_h \subset Sp_4(\mathbb{Z})$ is isomorphic to $SL_2(\mathcal{O})$. Let $\bar{h}$ and $\bar{k}$ denote the images in $L/2L$ and $\bar{\sigma}$ the image in $Sp_4(\mathbb{Z}/2)$. Since $\bar{h} = \bar{k}$, $\bar{\sigma} \in \Gamma_{\bar{h}}$. By strong approximation the reduction map $\Gamma_h \to \Gamma_{\bar{h}}$ is surjective. Hence we can write $\sigma = \sigma'\delta$ with $\sigma' \in \Gamma(2)$ and $\delta \in \Gamma_h$. It follows that $k = \sigma(h) = \sigma'\delta(h) = \sigma'(h)$.

LEMMA A.7. *Let $h \in L$ such that $\nu(h) \neq 0$ and $2Q'(h) = n$. Consider the element $v_n \in L$ with coordinates $(0, n, 0, -1, 0)$ with $n = 1 + 4r$. If h is primitive then h is in the $\Gamma(2)$ orbit of v_n. If $h = dh'$ with h' primitive then h is in the $\Gamma(2)$ orbit of dv_{n/d^2}.*

Proof. We have already seen that $\nu(h) \neq 0$ implies that $2Q'(h) \equiv 1 \bmod 4$, say $2Q'(h) = n = 1 + 4r$. Consider the element $v_n = (0, n, 0, -1, 0) = v_1 + 4re_{14}$. Then $Q'(v_n) = \frac{1}{2} + 2r = Q'(h)$. Assume first that h is primitive. Then Humbert's theorem implies that v_n and h are conjugate under $Sp_4(\mathbb{Z})$. Now h is in the $\Gamma(2)$ orbit of v_1 modulo $\mathscr{L}$ so in particular $h \equiv v_1 \bmod 2$, but also $v_n \equiv v_1 \bmod 2$ so the previous lemma gives that h and v_n are conjugate under $\Gamma(2)$. It follows that $\nu(h) = \nu(v_n)$.

Next consider the case where $h = dh'$ with h' primitive. Then $Q'(h) = d^2Q'(h')$ and $2Q'(h') \equiv 1 \bmod 4$. It follows that h' is in the $\Gamma(2)$ orbit of v_{n/d^2} and so h is in the orbit of dv_{n/d^2}.

Clearly $I(\phi, h) = I(\phi, dh') = I(\phi, h')$ and $\nu(h) = \nu(dh') = (-1/d)\,\nu(h') = (-1/d)\,\nu(v_{n/d^2}) = (-1/d)$. It follows that we have the formula for the Fourier coefficients:

$$c_n = \begin{cases} \sum_{d^2 \mid n} \left(\frac{-1}{d}\right) I(\phi, v_{n/d^2}) & \text{for } n \equiv 1 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the form $\Theta(\phi, \nu)(4z)$. This form is in $S_{5/2}(\Gamma_0(64))$. The formula for the Fourier coefficients shows that it is in fact in Kohnen's "+"-space,

$S^+_{5/2}(\Gamma_0(64))$, [Kohl]. It follows from the results of Niwa [Ni] that the Shimura lift is a cusp form of weight 4 for $\Gamma_0(32)$ with trivial character.

Next we consider the action of the Hecke operators. Let $A \in M_4(\mathbb{Z})$ with ${}^tA\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}A = r(A)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, i.e., $A \in GSp_4(\mathbb{Q}) \cap M_4(\mathbb{Z})$. Assume that $r(A)$ is odd and $A \equiv \begin{pmatrix} I & 0 \\ 0 & r(A)I \end{pmatrix} \bmod 4$. Write the double coset $\Gamma(2) \cdot A \cdot \Gamma(2) = \coprod \Gamma(2)A_i$, a finite union. Then the Hecke operator $T(\Gamma(2) \cdot A \cdot \Gamma(2))$ on $S_3(\Gamma(2), \xi)$ is given by $T(\Gamma(2) \cdot A \cdot \Gamma(2))(\phi) = r(A)^3 \sum \xi(\mu_i)\, \phi|_{A_i}$ where $\mu_i \equiv \begin{pmatrix} I & 0 \\ 0 & r(A)I \end{pmatrix} A_i^{-1} \bmod 4$. By abuse of notation we shall use $T(A)$ to denote this Hecke operator.

We want to compute the Fourier series of $\Theta(T(A)\phi, v)$ or equivalently the period integrals $I(T(A)\phi, v)$. We have

$$I(T(A)\phi, h) = r(A)^3 \int_{X_h/\Gamma(2)_h} \sum_i \xi(\mu_i)\, J(A_i, \tau)^{-3}\, \phi(A_i\tau)\, d\tau,$$

where $J(-, \tau)$ denotes the automorphy factor. The subgroup $\Gamma(2)_h \subset \Gamma(2)$ acts on the set of left cosets $\{\Gamma(2)A_i\}$ in the double coset $\Gamma(2) \cdot A \cdot \Gamma(2)$ by right multiplication and we consider a set of representatives for the orbits, $\{\Gamma(2)A_j\}$.

An element $g \in \Gamma(2)_h$ is in the stabilizer of a left coset $\Gamma(2)A_i$ if and only if $A_i g = \sigma A_i$ for some $\sigma \in \Gamma(2)$. Now $\sigma = A_i g A_i^{-1}$ implies that σ fixes $h_i = A_i h$ so the stabilizer of $\Gamma(2)A_i$ is equal to $\Gamma(2) \cap A_i^{-1}\Gamma(2)_{h_i}A_i$. Since ξ is trivial on $\Gamma(2)_h$ it follows that we can write the period integral

$$I(T(A)\phi, h) = r(A)^3 \int_{X_h/\Gamma(2)_h} \sum_j \xi(\mu_j) \times \sum_{g \in \Gamma(2)_h/\Gamma(2)_h \cap A_j^{-1}\Gamma(2)_h A_j} J(A_j g, \tau)^{-3}\, \phi(A_j g\tau)\, d\tau$$

where the first sum runs over the set of representatives. Clearly the inner sum is invariant under $\Gamma(2)_h$ and so the integral can be written

$$I(T(A)\tau, h) = r(A)^{-3} \sum_j \xi(\mu_j) \times \int_{X_h/\Gamma(2)_h \cap A_j^{-1}\Gamma(2)_{h_j}A_j} J(A_j, \tau)^{-3}\, \phi(A_j\tau)\, d\tau.$$

As τ runs over $X_h = G_h(\mathbb{R})\tau_0$ for $\tau_0 \in \mathfrak{S}_2$ some basepoint, $A_j\tau$ runs over $A_j G_h(\mathbb{R})_{\tau_0}$. We have $A_j G_h(\mathbb{R})\tau_0 = A_j G_h(\mathbb{R}) A_j^{-1} A_j \tau_0 = G_{h_j} A_j \tau_0$ changing variable $\tau' = A_j\tau$, using that $d(A_j\tau) = r(A_j)^{-3} J(A_j, \tau)^{-3}$

$d\tau = r(A)^{-3}\, J(A_j, \tau)^{-3}\, d\tau$ and the fact that the integral is independent of the choice of basepoint in $\mathfrak{S}_2$, we get

$$\begin{aligned} I(T(A)\phi, h) &= \sum_j \xi(\mu_j) \int_{X_{h_j}/\Gamma(2)_{h_j} \cap A_j \Gamma(2)_h A_j^{-1}} \phi(\tau)\, d\tau \\ &= \sum_j \xi(\mu_j)[\Gamma(2)_{h_j} : \Gamma(2)_{h_j} \cap A_j \Gamma(2)_h A_j^{-1}]\, I(\phi, h_j). \end{aligned}$$

Let p be an odd prime and let $T(p)$ be the Hecke operator associated to the double coset

$$\Gamma(2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \Gamma(2).$$

The following $p^3 + p^2 + p + 1$ matrices form a set of representatives for the left cosets in the double cosets

$$\text{(i)} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 1 & 0 & 2s & 2t \\ 0 & 1 & 2t & 2u \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad 0 \leqslant s, t, u \leqslant p-1,$$

$$\text{(iii)} \begin{pmatrix} p & 0 & 0 & 0 \\ -2t & 1 & 0 & 2u \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & p \end{pmatrix} \quad 0 \leqslant t, u \leqslant p-1,$$

$$\text{(iv)} \begin{pmatrix} 1 & 0 & 2r & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 0 \leqslant r \leqslant p-1.$$

In order to completely determine the action of the Hecke operator $T(p)$ on the period integrals, we would have to determine the orbits under $\Gamma(2)_{v_n}$. Luckily for our purposes it suffices to do this only for v_1.

Applying the matrices above to v_1 we get the following elements in L:

(i) pv_1

(ii) $(2s + 2u, p, 0, -p, 0), 0 \leqslant s, u, t \leqslant p-1$

(iii) $(2up, p^2, 2tp, -1-4t, 0), 0 \leqslant u, t \leqslant p-1$

(iv) $(pr, 1, 0, -p^2, 0), 0 \leqslant r \leqslant p-1.$

We see that type (ii) is $\Gamma(2)$ conjugate to pv_1 if and only if $p \mid 2s+2u$ and otherwise conjugate to v_{p^2}. Type (iii) is conjugate to pv_1 if and only if $p \mid 1+4t$ and otherwise conjugate to v_{p^2}. Type (iv) is conjugate to v_{p^2}. Two of the matrices above, A and B say, are in the same $\Gamma(2)_{v_1}$ orbit if and only if $B \in \Gamma(2) A \Gamma(2)_{v_1}$. In other words if we can find $M \in \Gamma(2)_{v_1}$ such that $AMB^{-1} \in \Gamma(2)$. Thus it is clear that if A and B are in the same $\Gamma(2)_{v_1}$ orbit then Av_1 is $\Gamma(2)$ conjugate to Bv_1. An easy but tedius computation, using that the isotropy group of v_1 is $SL_2(\mathbb{Z}[i])$, shows that there are two orbits:

(a) The orbit containing

$$A = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

consisting of the cosets with representatives

$$\begin{pmatrix} 1 & 0 & 2s & 2t \\ 0 & 1 & 2t & 2u \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \; p \mid 2s+2u$$

and

$$\begin{pmatrix} p & 0 & 0 & 0 \\ -2t & 1 & 0 & 2u \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & p \end{pmatrix}, \; p \mid 1+4t.$$

(b) The orbit containing

$$B = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

consisting of the remaining matrices.

It now follows that

$$\xi\left(\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^{-1} & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix} \bmod 4\right)$$
$$\times [\Gamma(2)_{v_1} : \Gamma(2)_{v_1} \cap A\Gamma(2)_{v_1} A^{-1}]\, I(\phi, v_1)$$

$$= +\xi\left(\begin{pmatrix} p^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bmod 4\right)$$

$$\times [\Gamma(2)_{v_{p^2}} : \Gamma(2)_{v_{p^2}} \cap B\Gamma(2)_{v_1} B^{-1}]\, I(\phi, v_{p^2}).$$

We have to compute the two indices $[\Gamma(2)_{v_1} : \Gamma(2)_{v_1} \cap A\Gamma(2)_{v_1} A^{-1}]$ and $[\Gamma(2)_{v_{p^2}} : \Gamma(2)_{v_{p^2}} \cap B\Gamma(2)_{v_1} B^{-1}]$. It is easy to see that the first index is equal to the index of the parabolic subgroup P of lower triangular matrices in $SL_2(\mathbb{Z}[i]/p)$.

If p does not split in $\mathbb{Z}[i]$, i.e., $(-1/p) = -1$, we have $SL_2(\mathbb{Z}[i]/p)/P = SL_2(\mathbb{F}_{p^2})/P = \mathbb{P}^1(\mathbb{F}_{p^2})$.

If p splits in $\mathbb{Z}[i]$, i.e., $(-1/p) = 1$, we have $SL_2([i]/p) = SL_2(\mathbb{F}_p) \times SL_2(\mathbb{F}_p)$ so $SL_2(\mathbb{Z}[i]/p)/P = \mathbb{P}^1(\mathbb{F}_p) \times \mathbb{P}^1(\mathbb{F}_p)$. Thus we have $[\Gamma(2)_{v_1} : \Gamma(2)_{v_1} \cap A\Gamma(2)_{v_1} A^{-1}] = p^2 + 1 + p(1 + (-1/p))$.

It is easy to see that $\Gamma(2)_{v_{p^2}} \subset B\Gamma(2)_{v_1} B^{-1}$ hence the second index is 1 and we get the following formula

$$I(T(p), v_1) = \left(p^2 + 1 + p\left(1 + \left(\frac{-1}{p}\right)\right)\right) I(\phi, v_1) + \left(\frac{-1}{p}\right) I(\phi, v_{p^2}).$$

Let $c_n(p)$ denote the nth Fourier coefficient of $\Theta(T(p)\phi, v)(4z)$. Using the formula above and the formula for the Fourier coefficients from Lemma A.7 we obtain the following formula:

$$c_1(p) = \left(p^2 + 1 + p\left(1 + \left(\frac{-1}{p}\right)\right)\right) c_1 + \left(\frac{-1}{p}\right)\left(c_{p^2} - \left(\frac{-1}{p}\right) c_1\right)$$

$$= \left(\frac{-1}{p}\right)(c_{p^2} + pc_1) + (p^2 + p)\, c_1.$$

Next recall that the action on the Fourier coefficients of Shimura's Hecke operator, $T_\chi(p^2)$ on the space $S_{k+1/2}(\Gamma_0(4N), \chi)$ is given by

$$a_n(p) = a_{np^2} + p^{k-1}\chi(p)\left(\frac{Nn}{p}\right) a_n + p^{2k-1}\chi(p^2)\, a_{n/p^2}.$$

Since ϕ is and eigenform, $\Theta(\phi, v)(4z)$ is an eigenform for the Shimura Hecke operators. Let $\{\lambda(p)\}$ denote the eigenvalues for ϕ and $\{\mu(p)\}$ the eigenvalues for $\Theta(\phi, v)(4z)$. We have shown that $c_1 \neq 0$ hence we get the following relation between the two sets of eigenvalues:

$$\lambda(p) = \left(\frac{-1}{p}\right)\mu(p) + (p^2 + p).$$

Consider the operator

$$T'(p^2) = T(p)^2 + T(p^2)$$
$$= T\begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (p^3 + p^2 + p)\, T\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

A long computation along the same lines as above gives the relation

$$\lambda(p)^2 - \lambda(p^2) = (p^2 + p)\left(\frac{-1}{p}\right)\mu(p) + 2p^3 + p^2$$

The local factor of the Andrianov–Evdokimov L-function of ϕ ([An, Ev1]) is given by

$$L_p(\phi, s)^{-1} = 1 - \lambda(p)\, p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^2)\, p^{-2s} - \lambda(p)\, p^{3-3s} + p^{6-4s}.$$

The relations above imply that

$$L_p(\phi, s) = (1 - p^{2-s})\left(1 - \left(\frac{-1}{p}\right)\mu(p)\, p^{-s} + p^{3-2s}\right)(1 - p^{1-s}).$$

Let f be the elliptic modular form associated to $\Theta(\phi, v)$ via the Shimura correspondence. Then $f \in S_4(\Gamma_0(32))$ and f is an eigenform with eigenvalues $\{\mu(p)\}$. The space $S_4(\Gamma_0(32))$ has dimension 5 and is spanned by the following eigenforms:

(i) The form $g(\tau) = \eta(2\tau)^4\eta(4\tau)^4$. This form is actually in $S_4(\Gamma_0(8))$.

(ii) The twist g^χ of g by the non-trivial character χ mod 4.

(iii) The old forms $g(4\tau)$, $g(2\tau)$ and $g^\chi(2\tau)$.

We have $c_1 \neq 0$ so f cannot be one of the old forms. To determine which of the two new-forms occur, it suffices to compute just one of the Hecke eigenvalues of ϕ. Using a computer we find $\lambda(11) = 88$ so $\mu(11) = 44$. The Fourier coefficient $a(11)$ of g is -44 and so it follows that the normalization of f is the twist g^χ.

The local factors in the L-function of f are given by

$$L_p(f, s)^{-1} = 1 - \mu(p)\, p^{-s} + p^{3-2s}$$

and so we obtain

$$L(\phi, s) = \zeta(s-1)\, L(g^{\chi}, \chi, s)\, \zeta(s-2) = \zeta(s-1)\, L(g, s)\, \zeta(s-2).$$

The modular group $Sp_4(\mathbb{Z})$ acts transitively on the 15 forms in $S_3(\Gamma(4))$ and so the L-functions of the other 14 forms are twists by characters of $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ of this L-function. Which twist occurs in each case can be determined by computing the Hecke eigenvalue $\lambda(11)$. In particular for the form considered in the first part of Section 2 we find that there is no twist so the Andrianov–Evdokimov L-function of this Siegel modular form is the same as the L-function for ϕ.

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