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Propositional modal logic

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The modes of truth

- The language of propositional logic allows us to represent complex English sentences formed by means of the connectives "and", "or", "not", and (if Grice or Lewis and Jackson are right) also some uses of the connectives "if" and "if and only if".
- Now, we want to represent complex English sentences formed by means of connectives like "it is possible that", "it is necessary that", "necessarily", "can", "must", etc.
- Expressions of this kind are called "modal", since traditionally necessity and possibility were thought of as *modes* in which a sentence can be true or false.

Truth functional connectives and modal connectives

- We saw that the connectives "and", "or", "not" (and perhaps also some occurrences of "if") are truth functional, namely the truth value of a complex sentence formed by means of these connectives is entirely determined by the truth values of the sentences they connect.
- On the other hand, expressions like "it is possible that", "it is necessary that", etc. are *not* truth functional.
- Let's remind ourselves of the reasons why the truth value of sentences of the form "it is possible that", "it is necessary that", etc. is not entirely determined by the truth value of A.

Possibility operators and truth functions

- Suppose Lea had a reasonable chance to win the election, but in the end Leo won. Suppose, moreover, that before the elections, I asserted (1) e (2):

(1) Lea will win the elections.

(2) It is possible that Lea will win the elections.

- Clearly, by asserting (1) I said something false, since Leo won. However, by asserting (2) I said something true, since Lea's victory was a possible outcome.
- On the other hand, it's clear that, if I assert (3) e (4), I said something false in both cases:

(3) Leo is a bachelor and Leo is not a bachelor.

(4) It is possible that Leo is a bachelor and Leo is not a bachelor.

- Thus, the truth value of sentences of the form ⌜it is possible that A⌝ is not entirely determined by the truth value of A. Indeed, as we have just seen, if "it is possible that" is applied to false sentence (1) the result is a true sentence (namely (2)), while if "it is possible that" is applied to false sentence (3) the result is a false sentence (namely (4)).

Necessity operators and truth functions

- A similar reasoning shows that "necessarily" is not a truth functional connective.
- Suppose that Leo never married, although he had come close to marry several times. So, sentence (5) is true, but sentence (6) is false, since Leo could have married, if things had gone differently:

(5) Leo is a bachelor.

(6) Necessarily, Leo is a bachelor.

- On the other hand, (7) and (8) are both true:

(7) If Leo is a bachelor, Leo is a bachelor.

(8) Necessarily, if Leo is a bachelor, Leo is a bachelor.

- Thus, the truth value of sentences of the form ⌜necessarily A⌝ is not entirely determined by the truth value of A. Indeed, if "necessarily" is applied to true sentence (5), the result (sentence (6)) is a false sentence, while if if "necessarily" is applied to true sentence (7) the result (sentence (8)) is a true sentence.

Modal connectives and truth-tables

- The above considerations show that it is not possible to describe the semantics of modal connectives by means of truth tables.
- Indeed, as we just saw, it is not possible to assign the same truth value to ⌜It is necessary that A⌝ in all cases in which A is true. And it is not possible to assign the same truth value to ⌜It is possible that A⌝ in all cases in which A is false:

A	Possibly A
V	*V*
F	?

A	Necessarily A
V	?
F	*F*

An authoritative suggestion

- In view of what we observed, how should we state the truth conditions of sentences formed by means of modal connectives?
- A suggestion comes from Leibniz. In a 1686 paper entitled "Necessary and contingent truths", Leibniz says:

  "[necessary truths] not only will they hold whilst the world remains, but they would have held even if God had created the world in another way."

Modal connectives as quantifiers over possible worlds

- The above passage by Leibniz suggests that a sentence of the form ⌜it is necessary that A⌝ is true exactly in case A is true not only in the real world, but *in every possible world*.
- Following a related intuition, we may say that a sentence of the form ⌜it is possible that A⌝ is true exactly in case A is true *in at least one possible world.*
- These ideas will be the starting point for a more precise formulation of the truth conditions of sentences of the form ⌜it is necessary that A⌝ and ⌜it is possible that A⌝
- Before we introduce a formal language which contains modal operators, however, two considerations are in order.

Possible worlds

- Spelling out the truth conditions of modal sentences by quantifying over possible worlds raises a natural question:
  - what is a possible world?
- Here is the answer given by D. Lewis.

D. Lewis on possible worlds

The world we live in is a very inclusive thing, Every stick and every stone you have ever seen is part of it. And so are you and I. And so are the planet Earth, the solar system, the entire Milky Way, the remote galaxies we see through telescopes, and (if there are such things) all the bits of empty space between the stars and galaxies. There is nothing so far away from us as not to be part of our world. Anything at any distance at all is to be included. Likewise the world is inclusive in time. No long-gone ancient Romans, no long-gone pterodactyls, no long-gone primordial clouds of plasma are too far in the past, nor are the dead dark stars too far in the future, to be part of this same world....
The way things are, at its most inclusive, means the way this entire world is. But things might have been different, in ever so many ways. This book of mine might have been finished on schedule. Or, had I not been such a commonsensical chap, I might be defending not only a plurality of possible worlds, but also a plurality of impossible worlds, whereof you speak truly by contradicting yourself. Or I might not have existed at all... Or there might never have been any people. Or the physical constants might have had somewhat different values, incompatible with the emergence of life. Or there might have been altogether different laws of nature; and instead of electrons and quarks, there might have been alien particles, without charge or mass or spin but with alien physical properties that nothing in this world shares. There are ever so many ways that a world might be; and one of these many ways is the way that this world is.
D. Lewis, *On the Plurality of Worlds*, 1986, pp. 1-2.

The debate on the nature of possible worlds

- Lewis's idea is that there are many different possible worlds, concrete entities like our world, but distinct from it. This thesis is called *modal realism*.
- However, the claim that modal connectives, from a semantic standpoint, are quantifiers over possible worlds says nothing about the nature of the entities over which these operators quantify. In particular, the claim that modal connectives are quantifiers over worlds does not commit us to accept that they are concrete entities, like the real world, but distinct from it.
- An advocate of the thesis that modal connectives are quantifiers over possible worlds might insist that there is only one world, the actual world, and that the entities over which modal connectives quantify are also part of this world (for example, one might argue that they are abstract entities, like sets of sentences, properties, etc.). This thesis is called *modal actualism*.
- The proper place for the debate on the nature of possible world is metaphysics course, not a course on logic and language. So, I won't pursue the issue further.

Polysemy of modal expressions

- The second consideration, which has more direct consequences for us, concerns the fact that modal expressions come with many different senses.
- Let's see some examples.

Metaphysical necessity and possibility

- Sentences (9) and (10) illustrate the sense of "possibility" called "metaphysical possibility":

(9) It is not possible that Socrates is a prime number.

(10) It is not possible that it's raining and it's not raining.

- Sentences (11) and (12) illustrate the sense of "necessity" called "metaphysical necessity" (or "broad logical necessity", to use a term by A. Plantinga):

(11) Necessarily, Socrates is not a prime number.

(12) Necessarily, if it's raining, it's raining.

M. Jubien on metaphysical possibility

. . . metaphysical possibility is an ordinary concept, like truth. It is the broadest (or most inclusive) notion of possibility that we employ in thinking about how the world might have been. [. . .] The metaphysical notion of possibility may be illuminated as follows. Imagine that God truly exists and, as traditionally conceived, he is "all-powerful". This means he can bring about any possible *situation. He can cause a purple cow to come suddenly into being in the Oval Office of the White House. But he cannot cause any cow to be entirely purple and entirely green. He can cause all physical matter or any fragment of physical matter to cease existing instantaneously. But he cannot cause a specific fragment of matter to be entirely located at separated places at the same time. He can create a perfect duplicate of our entire solar system, right down to every subatomic particle, somewhere in deep space, or even in some unconnected spacetime. But he cannot bring it about that* your duplicate *in that solar system is* you.

The mere fact that we can conceive that God could, if he chose, produce these remarkable states of affairs, shows that we already think of them as, in an appropriate sense, possible. So their possibility does not actually depend on the existence of God, nor on any such possibility ever being actual. This familiar concept of possibility is the metaphysical one we have been seeking. Of course, as with any ordinary philosophical concept, our grasp of it is not so certain as to rule out disagreement over specific cases. (M. Jubien, *Contemporary metaphysics*, 1997, pp. 131-3).

Deontic necessity and possibility

- Let's now consider sentences (13) and (14):

(13) It is necessary that young people give up their seat for the elderly.

(14) It is possible to speak in a low voice.

- When we assert (13), presumably we mean that it is our obligation to make way for the elders. And when we assert (14), presumably we mean that our obligations allow us to speak in a low voice.
- When by using words like "necessary" and "possible" we mean "necessary in view of our obligations" and "possible in view of our obligations", the relevant notions of necessity and possibility are called "*deontic*".

Physical necessity and possibility

- Now consider sentences (15) and (16):

(15) It is not possible to travel faster than light.

(16) Necessarily, if the litmus paper is red, the solution is acidic.

- If we assert (15), presumably we mean that traveling faster than light is not compatible with our laws of nature. And if we assert (16) presumably we mean that our laws of nature require that the solution is acidic if the litmus paper is red.
- When by using words like "necessary" and "possible" we mean "necessary in view of our laws of nature" and "possible in view of our laws of nature", the relevant notions of necessity and possibility are called "*physical*".

Epistemic necessity and possibility

- Now suppose you are sampling a wine and, after careful thought, you say:

(17) This wine must have aged in *barrique*.

- What you mean by (17) is that, in view of the knowledge you acquired by sampling the wine, you conclude the wine was aged in *barrique*.
- Now suppose you assert (18):

(18) It's possible that Peter has already left.

- In this case, presumably what you mean is that, given what you know, you cannot exclude that Peter has already left.
- When by "necessary" e "possible" we mean "necessary in view of what we know" and "possible in view of what we know", the relevant notions of necessity and possibility are called "*epistemic*".

Historical necessity and possibility

- Finally, consider sentence (19):

(19) In 1932 it was possible for Great Britain to avoid war with Germany; but in 1937 it was necessary to go to war (Thomason 2002)

- Presumably, if someone utters (19), she means that while the course of events till 1932 was compatible with a pacific solution, the situation in 1937 excluded the possibility of a pacific solution.
- When by "necessary" e "possible" we mean "necessary in view of the course of events up to a given moment" and "possible in view of the course of events up to a given moment", the relevant notions of necessity and possibility are called "*historical*".

Validity and types of necessity

- We have seen that English modal connectives may occur with different senses, may express different notions of necessity and possibility.
- Recall that our aim is to single out the *argument forms that ensure that an argument is valid* (representing English sentences in a logical language is a way of pursuing this goal).
- For this reason, it is important to be aware that different senses of "necessary" e "possible" may generate different valid argument forms.
- Let's see an example that illustrates this point.

A valid argument form

metaphysical necessity

- Consider sentence (20):

(20) Necessarily, Socrates is identical to Socrates.

- Sentence (20) expresses a metaphysical necessity.
- If "necessarily" expresses a metaphysical necessity, arguments of the form ⌜Necessarily A. Therefore, A⌝ are valid.
- Indeed, (21) follows from (20):

(21) Socrates is identical to Socrates.

A valid argument form

physical necessity

- Consider sentence (16) again:

(16) Necessarily, if the litmus paper is red, the solution is acidic.

- Sentence (16) expresses a physical necessity.
- If "necessary" expresses a physical necessity, again arguments of the form ⌜Necessarily A. Therefore, A⌝ are valid.
- Indeed, if our physical laws require that, if the litmus paper is red, the solution is acidic, (22) must be true in our world. Thus, (22) follows from (16):

(22) If the litmus paper is red, the solution is acidic.

An invalid argument form

deontic necessity

- Now consider sentence (13) again:

(13) It is necessary that young people give up their seat for the elderly.

- Sentence (13) expresses a deontic necessity.
- If "necessary" expresses a deontic necessity, arguments of the form ⌜Necessarily A. Therefore, A⌝ need not be valid.
- Indeed, (23) does not follow from (13):

(23) Young people give up their seat for the elderly.

- Sentence (13) says that our obligations require young people to give up their seat for the elderly. But it may very well happen that in the actual world people do not behave according to their obligations.

The importance of keeping things apart

- English modal connectives may express different kinds necessity and possibility, and these different senses of English modal connectives may give rise to different valid argument forms.
- Therefore, when we represent English modal connectives in a logical language we must be careful in distinguishing these different senses.
- If we don't, we won't be able to capture the valid argument forms associated to the different senses of the modal connectives.

Symbols for metaphysical modalities

- A way to represent the different senses of modal connectives in a logical language is to introduce a different symbol for each sense.
- Here we will focus on the metaphysical notions of necessity and possibility.
- To represent the metaphysical sense of "it is possible that" and "it is necessary that", we will use, respectively, the symbols "$\Diamond$" e "$\Box$".

A problem

- A problem that arises in describing the semantics of modal connectives "$\Box$" e "$\Diamond$" (which are meant to represent the metaphysical sense of "possible" and "necessary") is that there is no consensus among philosophers on the valid argument forms for the metaphysical sense of "possible" and " necessary".
- For example, Plantinga (1974) and many other philosophers, think that, if "possible" and "necessary" are understood metaphysically, principles 1-3 are valid:
  1. What is necessary is true.
  2. What is necessary is necessarily necessary.
  3. What is possible is necessarily possible.
- However, other philosophers, like Salmon (1989), Bonevac (2003) and Soames (2010), disagree. In particular, principles 2 and 3 are controversial.
- So, how should we proceed in stating the semantics of "$\Box$" and "$\Diamond$"?

Accessibility relations

- A way to provide alternative semantics for "$\Box$" e "$\Diamond$", which reflect different hypotheses on the valid argument forms for metaphysical necessity and possibility, is to make use of *accessibility relations*.
- Recall that, in our initial formulation based on Leibniz's passage, we said that ⌜it is necessary that A⌝ is true if and only if A is true in every possible world, and ⌜it is possible that A⌝ is true if and only if A is true in at least one possible world.
- Our formulation for "$\Box$" e "$\Diamond$" will have a slightly different form:
  - $\ulcorner\Box\varphi\urcorner$ is true in a world w if and only if φ is true in every possible world *accessible from w*;
  - $\ulcorner\Diamond\varphi\urcorner$ is true in a world w if and only if φ is true in at least one possible world *accessible from w*.

Relative possibility and necessity

- According to our revised formulation for the semantics of "$\Box$" e "$\Diamond$", what is necessary or possible may depend, in principle, from the world of evaluation: the truth of $\ulcorner\Box\varphi\urcorner$ and $\ulcorner\Diamond\varphi\urcorner$ at a world w depends on which worlds are accessible from w.
- In this sense, our revised formulation of the the semantics of "$\Box$" e "$\Diamond$", unlike the Leibnizian one for "it is necessary that" and "it is possible that", makes use of *relative* notions of necessity and possibility.

How to represent the disagreement

- The disagreement on the valid argument forms for metaphysical modalities may now be stated as a disagreement on the *formal properties of the accessibility relation*.
- For example, an accessibility relation R, like any other relation, may have or fail to have properties of these kinds:
  - universality: for every w, w', wRw';
  - reflexivity: for every w, wRw;
  - symmetry: for every w, w', if wRw', then w'Rw;
  - transitivity: for every w, w', w'', if wRw' and w'Rw'', then wRw'';
  - extendibility: por every w, there is a w' such that wRw';
  (we read *wRw'* as 'world w' is accessible from w').
- In stating the semantics of "$\Box$" and "$\Diamond$", one may assume that the accessibility relation meets some formal properties and does not meet other formal properties.
- The different properties one chooses to impose on the accessibility relation give rise to different valid argument forms.

Taking sides

- For our purposes here, we will assume Plantinga's view concerning the principles that hold true for the metaphysical sense of "possible" and " necessary":
  1. What is necessary is true.
  2. What is necessary is necessarily necessary.
  3. What is possible is necessarily possible.
- We are going to present a language, called LS5, that validates these principles.

The language LS5

symbols

- An infinite number of propositional letters: p_1 p_2 p_3 p_4 ...
- The connectives: $\wedge$ $\vee$ $\supset$ $\equiv$ $\sim$
- The parentheses: ( )
- The necessity and possibility operators: $\Box$ $\Diamond$

The language LS5

well-formed formulae

(a) The propositional letters are well-formed formulae of LS5 (the *atomic formulae*).
If φ e ψ are well-formed formulae of LS5, then:
(b) $\sim \varphi$ is a well-formed formula of LS5,
(c) $(\varphi \wedge \psi)$ is a well-formed formula of LS5,
(d) $(\varphi \vee \psi)$ is a well-formed formula of LS5,
(e) $(\varphi \supset \psi)$ is a well-formed formula of LS5,
(f) $(\varphi \equiv \psi)$ is a well-formed formula of LS5,
(g) $\Box\varphi$ is a well-formed formula of LS5,
(h) $\Diamond\varphi$ is a well-formed formula of LS5.
(i) Nothing else is a well-formed formula of LS5.

(Convention: we may leave the parentheses out, when it does not create ambiguities).

Truth at a world

- A fundamental ingredient of the semantics of LS5, in addition to possible worlds and the accessibility relation, is the notion of truth at a world (which we have already introduced informally in the above discussion).
- Unlike the valuations for LP, which assign truth value to the well-formed formulae, the valuations for LS5 assign truth value to the well-formed formulae *at a world*.
- Let's see how.

The language LS5

models

A model for LS5 is a triple <W, R, ν>, where

1. W is a non-empty set of possible worlds.
2. R is a *universal* binary relation between elements of W, the accessibility relation. (Recall: an accessibility relation is universal, according to the definition given above, if every world is accessible from every world).
3. ν is a function that assigns a truth value at a world to the well-formed formulae of LS5 in the following way:

for every world w in W,

(a) if φ is a propositional letter of LS5, then $\nu(\varphi, w) \in \{0, 1\}$;
if φ and ψ are well-formed formulae of LS5, then:

(b) $\nu(\sim \varphi$, w)=1 if $\nu(\varphi$, w)=0, otherwise $\nu(\sim \varphi$, w)=0;

(c) $\nu(\varphi \wedge \psi$, w)=1 if $\nu(\varphi$, w)=1 and $\nu(\psi$, w)=1, otherwise $\nu(\varphi \wedge \psi$, w)=0;

(d) $\nu(\varphi \vee \psi$, w)=1 if it is not the case that $\nu(\varphi$, w)=0 and $\nu(\psi$, w)=0, otherwise $\nu(\varphi \vee \psi$, w)=0;

(e) $\nu(\varphi \supset \psi$, w)=1 if it is not the case that $\nu(\varphi$, w)=1 and $\nu(\psi$, w)=0, otherwise $\nu(\varphi \supset \psi$, w)=0;

(f) $\nu(\varphi \equiv \psi$, w)=1 if $\nu(\varphi$, w)=$\nu(\psi$, w), otherwise $\nu(\varphi \equiv \psi$, w)=0.

(g) $\nu(\Box\varphi$, w)=1 if for every w' in W such that wRw', $\nu(\varphi$, w')=1, otherwise $\nu(\Box\varphi$, w)=0;

(h) $\nu(\Diamond\varphi$, w)=1 if there is at least a w' in W such that wRw' and $\nu(\varphi$, w')=1, otherwise $\nu(\Diamond\varphi$, w)=0.

("$\nu(\varphi$, w)=1/0" is read "the truth value of φ at the world w is 1/0").

Validity in LS5

- An argument in LS5 with premises $\varphi_1, \ldots, \varphi_n$ and conclusion ψ is *valid* in LS5 if and only if there is no model <W, R, ν> of LS5 and no world w in W such that $\nu(\varphi_1, w) = 1, \ldots, \nu(\varphi_n, w) = 1$ and $\nu(\psi, w) = 0$.
- If an argument is valid in LS5, we will say that its premises *logically imply* its conclusion in LS5.
- In symbols, when an argument in LS5 is valid, we shall write:

$$\{\varphi_1, \ldots, \varphi_n\} \models_{LS5} \psi$$

- A well-formed formula φ of LS5 is valid ($\models_{LS5} \varphi$) if and only if there is no model <W, R, ν> of LS5 and no world w in W such that $\nu(\varphi$, w)=0.

Historical note

Kripke models

- This way of spelling out the semantics of modal connectives, by means of models consisting of a set of worlds, an accessibility relation, and a function which assigns a truth value at a world to the well-formed formulae, is due to Kripke (1963).
- However, the idea of using an accessibility relation between worlds for the semantics of modal expressions occurred to several authors more or less at the same time (Geach, Prior, Hintikka, Kanger, Montague e Kripke).
- (A history of the development of modal logic from the end of the 19th century to today is provided in Goldblatt 2005).

Properties di LS5

- Let's now consider some properties of the language LS5.

Properties of LS5

the necessitation rule

- Notice that, if a formula φ is valid in LS5, the necessitation φ is also valid: in symbols, if $\models_{LS5} \varphi$, then $\models_{LS5} \Box\varphi$.
- We may convince ourselves that this is the case by reasoning thus:
  - If φ is valid in LS5, then for every model <W, R, ν> of LS5 and every world w in W, $\nu(\varphi$, w)=1;
  - Thus, for every model <W, R, ν> of LS5 and every world w in W, $\nu(\varphi$, w')=1 for every world w' in W such that wRw';
  - Thus, for every model <W, R, ν> of LS5 and every world w in W, $\nu(\Box\varphi$, w)=1, namely $\models_{LS5} \Box\varphi$.

Properties of LS5

equivalence of $\ulcorner{\sim}\Diamond\varphi\urcorner$ and $\ulcorner\Box{\sim}\varphi\urcorner$

- Notice that ${\sim}\Diamond\varphi \models_{LS5} \Box{\sim}\varphi$, for every formula φ.
  Indeed:
  - suppose that <W, R, ν> is a model of LS5 and w is a world in W such that $\nu({\sim}\Diamond\varphi$, w)=1;
  - thus, $\nu(\Diamond\varphi$, w)=0;
  - thus, there is no world w' in W such that wRw' and $\nu(\varphi$, w')=1;
  - thus, for every world w' in W such that wRw', $\nu(\varphi$, w')=0;
  - thus, for every world w' in W such that wRw', $\nu({\sim}\varphi$, w')=1;
  - thus, $\nu(\Box{\sim}\varphi$, w)=1;
  - thus, there is no model <W, R, ν> di LS5 and no world w in W such that $\nu({\sim}\Diamond\varphi$, w)=1 and $\nu(\Box{\sim}\varphi$, w)=0, namely ${\sim}\Diamond\varphi \models_{LS5} \Box{\sim}\varphi$.
- In a similar way, one may show that $\Box{\sim}\varphi \models_{LS5} {\sim}\Diamond\varphi$.

Properties of LS5

equivalence of $\ulcorner{\sim}\Box\varphi\urcorner$ and $\ulcorner\Diamond{\sim}\varphi\urcorner$

- Note that ${\sim}\Box\varphi \models_{LS5} \Diamond{\sim}\varphi$, for every formula φ.
  Indeed:
  - Suppose that <W, R, ν> is a model of LS5 and w is a world in W such that $\nu({\sim}\Box\varphi$, w)=1;
  - thus, $\nu(\Box\varphi$, w)=0;
  - thus, there is a world w' in W such that wRw' and $\nu(\varphi$, w')=0;
  - thus, there is a world w' in W such that wRw' and $\nu({\sim}\varphi$, w')=1;
  - thus, $\nu(\Diamond{\sim}\varphi$, w)=1;
  - thus, there is no model <W, R, ν> di LS5 and no world w in W such that $\nu({\sim}\Box\varphi$, w)=1 and $\nu(\Diamond{\sim}\varphi$, w)=0, namely ${\sim}\Box\varphi \models_{LS5} \Diamond{\sim}\varphi$.
- With a similar reasoning one may show that $\Diamond{\sim}\varphi \models_{LS5} {\sim}\Box\varphi$.

Properties of LS5

equivalence of $\ulcorner\Box\varphi\urcorner$ e $\ulcorner\sim\Diamond\sim\varphi\urcorner$

- Note that $\Box\varphi \models_{LS5} \sim\Diamond\sim\varphi$, for every formula φ. Indeed:
  - suppose instead that $\Box\varphi \not\models_{LS5} \sim\Diamond\sim\varphi$;
  - thus, there is a model <W, R, ν> of LS5 and a world w in W such that $\nu(\Box\varphi$, w)=1 and $\nu(\sim\Diamond\sim\varphi$, w)=0;
  - thus, $\nu(\Diamond\sim\varphi$, w)=1;
  - thus $\nu(\sim\Box\varphi$, w)=1 (in virtue of the above established equivalence);
  - thus, $\nu(\Box\varphi$, w)=0, contrary to the hypothesis.
- With a similar reasoning one may show that $\sim\Diamond\sim\varphi \models_{LS5} \Box\varphi$.

Properties of LS5

equivalence of $\ulcorner\Diamond\varphi\urcorner$ and $\ulcorner\sim\Box\sim\varphi\urcorner$

- Note that $\Diamond\varphi \models_{LS5} \sim\Box\sim\varphi$, fo every formula φ. Indeed:
  - suppose instead that $\Diamond\varphi \not\models_{LS5} \sim\Box\sim\varphi$;
  - thus, there is a model <W, R, ν> of LS5 and a world w in W such that $\nu(\Diamond\varphi$, w)=1 and $\nu(\sim\Box\sim\varphi$, w)=0;
  - thus, $\nu(\Box\sim\varphi$, w)=1;
  - thus $\nu(\sim\Diamond\varphi$, w)=1;
  - thus, $\nu(\Diamond\varphi$, w)=0, contrary to the hypothesis.
- With a similar reasoning one may show that $\sim\Box\sim\varphi \models_{LS5} \Diamond\varphi$.

The principles characterizing metaphysical necessity

Plantinga

- Now we will show that LS5 validates the principles that, according to Plantinga, characterize metaphysical necessity:
  1. What is necessary is true.
  2. What is necessary is necessarily necessary.
  3. What is possible is necessarily possible.

What is necessary is true

- The accessibility relation of the models of LS5 is universal, namely every world is accessible from every world.
- Thus, in the models of LS5 every world is also accessible from itself.
- One consequence of this is that $\ulcorner\Box\varphi \supset \varphi\urcorner$ is valid in LS5, for every formula φ.
- Indeed, suppose that, there is a formula φ such that $\ulcorner\Box\varphi \supset \varphi\urcorner$ is not valid in LS5. In this case, there is a model for LS5 such that $\ulcorner\Box\varphi \supset \varphi\urcorner$ is false in some world of the model. Suppose w is such a world. Then, $\ulcorner\Box\varphi\urcorner$ is true in w and φ is false in w. Since $\ulcorner\Box\varphi\urcorner$ is true in w, φ is true in all the worlds accessible from w. Since w is accessible from w, φ is true in w, contradicting the above conclusion.

What is necessary is necessarily necessary

- It may be shown that $\ulcorner \Box\varphi \supset \Box\Box\varphi \urcorner$ is valid in LS5, for every formula φ.
- Indeed, suppose that there is a formula φ such that $\ulcorner \Box\varphi \supset \Box\Box\varphi \urcorner$ is not valid in in LS5. Thus, there is a model for LS5 such that $\ulcorner \Box\varphi \urcorner$ is true and $\ulcorner \Box\Box\varphi \urcorner$ is false in some world of the model. Suppose w is such a world. Thus, φ is true in every world accessible from w (since $\ulcorner \Box\varphi \urcorner$ is true in w). Furthermore, for some world w' accessible from w, $\ulcorner \Box\varphi \urcorner$ is false in w' (since $\ulcorner \Box\Box\varphi \urcorner$ is false in w). Thus, for some w" accessible from w', φ is false in w". But R is universal, and thus w" is accessible from w. Thus, since φ is true in every world accessible from w, φ is also true in w", contradicting the above conclusion.

What is possible is necessarily possible

- It may be shown that $\ulcorner \Diamond\varphi \supset \Box\Diamond\varphi \urcorner$ is valid in LS5, for every formula φ.
- Indeed, suppose that there is a formula φ such that $\ulcorner \Diamond\varphi \supset \Box\Diamond\varphi \urcorner$ is not valid in in LS5. Thus, there is a model for LS5 such that $\ulcorner \Diamond\varphi \urcorner$ is true and $\ulcorner \Box\Diamond\varphi \urcorner$ is false in some world of the model. Suppose w is such a world. Thus, there is a world w' accessible from w such that φ is true in w' (since $\ulcorner \Diamond\varphi \urcorner$ is true in w). Furthermore, for some world w" accessible from w, $\ulcorner \Diamond\varphi \urcorner$ is false in w" (since $\ulcorner \Box\Diamond\varphi \urcorner$ is false in w). Thus, φ is false in every world accessible from w" (since $\ulcorner \Diamond\varphi \urcorner$ is false in w"). But R is universal, and thus w' is accessible from w". Thus, φ is false in w', contradicting the above conclusion.

An equivalent formulation of the semantics of LS5

- It is possible to show (although we won't do it here) that the arguments that turn out to be valid if the accessibility relation is universal are exactly the arguments that turn out to be valid if the accessibility relation is reflexive, transitive and symmetric.
- For this reason, the semantics of LS5 is sometimes stated by assuming that the accessibility relation is reflexive, transitive and symmetric.

Iteration of modalities in LS5

- Let's conclude, by mentioning a further property of LS5 (without proving it).
- The definition of well-formed formulae of LS5 allows one to build formulae which contain consecutive sequences of modal operators.
- For example, the following are well-formed formulae of LS5:
  - (a) $\Diamond\Box\Box\Diamond\Box\Diamond\Diamond\Box p$
  - (b) $\Diamond\Box\Box\Diamond\Box\Diamond\Diamond(p \supset \Diamond q)$
  - (c) $\Box\Diamond\Box\Box\Diamond\Box\Diamond\Diamond\sim\Diamond q$
- In LS5, if we delete all the consecutive modal operators in these formulas except the last one (except the blue one in the formulae) we obtain an equivalent formula.
- Namely, in LS5 formulae (a)-(c) are equivalent to (a')-(c'):
  - (a') $\Box p$
  - (b') $\Diamond(p \supset \Diamond q)$
  - (c') $\Diamond\sim\Diamond q$

Natural deduction for propositional modal logic

- Now we are going to introduce a natural deduction system for LS5, which we shall call S5(NAT).
- The system is based on Gettier (1982) e Salmon (1994).

The system S5(NAT)

- The system S5(NAT) consists of these rules:
  - all the rules of LP(NAT);
  - the inference rules $\Box$E, $\Diamond$I, MN + the boxing and canceling rules specific to S5(NAT).

Inference rules $\Box$E, $\Diamond$I, MN

$\Box$E $\dfrac{\Box\varphi}{\varphi}$	$\Diamond$I $\dfrac{\varphi}{\Diamond\varphi}$		
MN $\dfrac{\sim\Box\varphi}{\Diamond\sim\varphi}$	$\dfrac{\sim\Diamond\varphi}{\Box\sim\varphi}$	$\dfrac{\Box\sim\varphi}{\sim\Diamond\varphi}$	$\dfrac{\Diamond\sim\varphi}{\sim\Box\varphi}$

Boxing and canceling rules of S5(NAT)

The boxing and canceling rules specific to S5(NAT) are the following:

$\Box$I

~~Prova:~~ $\Box\varphi$ $\Box$I

. nessuna ass.
.
.
φ

$\Diamond$E

i. $\Diamond\varphi$
.
.
k. ~~Prova:~~ $\Diamond\psi$ $\Diamond$E, i
l. φ Ass
.
.
.
m. ψ

– provided that line i is available and all references outside of Prova are by rule R to available lines of the form $\ulcorner\Box\chi\urcorner$ or $\ulcorner\Diamond\chi\urcorner$.

Comment on rule $\Box$I

- The introduction of necessity rule ($\Box$I) provides a way of proving formulae of the form $\ulcorner \Box\varphi \urcorner$.
- If we want to prove a formula $\ulcorner \Box\varphi \urcorner$ by using this rule, the only lines outside the proof we can use are available lines of the form $\ulcorner \Box\chi \urcorner$ or $\ulcorner \Diamond\chi \urcorner$.
- If we can derive φ either by importing available lines of these forms or by importing nothing, we have proved $\ulcorner \Box\varphi \urcorner$.
- This way of proving sentences of the form $\ulcorner \Box\varphi \urcorner$ is based on the idea that what is derivable from necessary statements (or from no premise at all) is also necessary.
- (Recall that the semantics of LS5 validates the principle that what is possible is necessarily possible. So, an available line outside the proof of the form $\ulcorner \Diamond\chi \urcorner$ is, in this sense, a necessary statement).

Comment on rule $\Diamond$E

- The possibility elimination rule ($\Diamond$E) provides a way of proving formulae of the form $\ulcorner \Diamond\psi \urcorner$.
- If a line of the form $\ulcorner \Diamond\varphi \urcorner$ is available, we may prove $\ulcorner \Diamond\psi \urcorner$ via the rule of possibility elimination by deriving ψ from assumption φ.
- This way of proving sentences of the form $\ulcorner \Diamond\psi \urcorner$ is based on the idea that what is derivable from possible statements is also possible.

Completeness and correctness

- It may be shown (we won't do it here) that a conclusion is derivable from a set of premises in the system S5(NAT) exactly in case the argument consisting of the premises and the conclusion is valid in LS5.
- In symbols:

  $\{\varphi_1, \ldots, \varphi_n\} \vdash_{S5(NAT)} \psi$ if and only if
  $\{\varphi_1, \ldots, \varphi_n\} \models_{LS5} \psi$.

- (As a particular case, it may be shown that $\vdash_{S5(NAT)} \psi$ if and only if $\models_{LS5} \psi$).