Exercises on derivations in propositional logic

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Primo esercizio

First examine the following derivations to get an idea of how the system LP(NAT) works. Suppose you want to prove (1):

(1) $p \vdash_{LP(NAT)} q \supset p$

A possible derivation is this:

1.	p	P
2.	~~Prova~~: $q \supset p$	$\supset$I
3.	q	Ass
4.	p	R, 1

Now suppose you want to prove (2):

(2) $\sim p \vdash_{LP(NAT)} p \supset q$

A possible derivation is this:

1.	$\sim p$	P
2.	~~Prova~~: $p \supset q$	$\supset$I
3.	p	Ass
4.	~~Prova~~: q	$\sim$E
5.	$\sim q$	Ass
6.	p	R, 3
7.	$\sim p$	R, 1

Finally, suppose you want to prove (3):

(3) $p \supset \sim q, q \vdash_{LP(NAT)} \sim p$

A possible derivation is this:

1.	$p \supset\sim q$	P
2.	q	P
3.	~~Prova~~: $\sim p$	$\sim$I
4.	p	Ass
5.	$\sim q$	$\supset$E, 1, 4
6.	q	R, 2

In doing derivations in LP(NAT), you should pay attention to the condition (Rule for inferences) by which you can apply an inference rule to a line only if the line is *available* (namely, the line is not boxed and contains no uncanceled 'Prova'). If you do not obey this condition, you can prove invalid formulae. For example, if you do not obey this condition, you can prove formulae like "$\sim p$" e "$p \wedge \sim p$":

1.	~~Prova~~: $\sim p$	DD
2.	~~Prova~~: $(p \wedge \sim p) \supset p$	$\supset$I
3.	$p \wedge \sim p$	Ass
4.	p	$\wedge$E, 3
5.	$\sim p$	$\wedge$E, 3

1.	~~Prova~~: $p \wedge \sim p$	DD
2.	p	$\wedge$E, 1
3.	$\sim p$	$\wedge$E, 1
4.	$p \wedge \sim p$	$\wedge$I, 2, 3

These are not derivations in LP(NAT), since line 5 in the first derivation and lines 2-3 in the second derivation apply inference rules to lines that are not available (indeed, when in line 5 of the first derivation rule $\wedge$E is applied to line 3, line 3 is already boxed. And when in lines 3 aand 4 of the second derivation rule $\wedge$E is applied to line 1, line 1 contains the word "prova" uncanceled.

Ok, now you go on by yourself. Prove the following claims:

(4)
a. $\sim (p \supset q) \vdash_{LP(NAT)} \sim q$
b. $p \supset q, q \supset r \vdash_{LP(NAT)} p \supset r$ (transitivity)
c. $\vdash_{LP(NAT)} (p \supset q) \vee (q \supset p)$
d. $\sim (p \supset q) \vdash_{LP(NAT)} p$
e. $(p \wedge q) \supset r, p \vdash_{LP(NAT)} q \supset r$
f. $\vdash_{LP(NAT)} \sim (p \wedge \sim p)$ (non contradiction)
g. $\vdash_{LP(NAT)} p \vee \sim p$ (excluded middle)
h. $p \supset q, \sim q \vdash_{LP(NAT)} \sim p$ (*modus tollens*)
i. $q \supset r \vdash_{LP(NAT)} (p \wedge q) \supset r$ (strengthening of the antecedent)

Second exercise (derived rules)

We said that every valid argument in LP is such that the conclusion is derivable from the premises in LP(NAT). Moreover, we said that every argument in LP whose conclusion is derivable from the premises in LP(NAT) is valid in LP. So there is no need to add more inference rules. However, it is convenient to add some rules which, although they do not allow us to prove anything new, sometimes make the derivations shorter. Here they are (the double line indicates that the rule can be applied in both directions):

$\vee\supset$ (prova per casi) $\quad \dfrac{\varphi \vee \psi \quad \varphi \supset \xi \quad \psi \supset \xi}{\xi}$	$\sim\wedge$ (DeMorgan) $\quad \dfrac{\sim(\varphi \wedge \psi)}{\sim\varphi \vee \sim\psi}$
$\sim\vee$ (DeMorgan) $\quad \dfrac{\sim(\varphi \vee \psi)}{\sim\varphi \wedge \sim\psi}$	$\sim\supset$ $\quad \dfrac{\sim(\varphi \supset \psi)}{\varphi \wedge \sim\psi}$
$\sim\equiv$ $\quad \dfrac{\sim(\varphi \equiv \psi)}{\sim\varphi \equiv \psi} \quad \dfrac{\sim(\varphi \equiv \psi)}{\varphi \equiv \sim\psi}$	$\supset\vee$ $\quad \dfrac{\varphi \supset \psi}{\sim\varphi \vee \psi}$
$\wedge$A $\quad \dfrac{(\varphi \wedge \psi) \wedge \xi}{\varphi \wedge (\psi \wedge \xi)}$	$\vee$A $\quad \dfrac{(\varphi \vee \psi) \vee \xi}{\varphi \vee (\psi \vee \xi)}$
$\supset$E* $\quad \dfrac{\varphi \supset \psi \quad \sim\psi}{\sim\varphi}$	$\equiv$E* $\quad \dfrac{\varphi \equiv \psi \quad \sim\varphi}{\sim\psi} \quad \dfrac{\varphi \equiv \psi \quad \sim\psi}{\sim\varphi}$
! (contraddizione) $\quad \dfrac{\varphi \quad \sim\varphi}{\psi}$	

Prove the following claim first without making use of any derived rule and then by making use of a derived rule:

(5) a. $p \vee q, p \supset r, q \supset r \vdash_{LP(NAT)} r$